QUANTUM INFORMATION AND COMPUTATION

"Information is physical"

CBIT (classical bit) → QUBIT (quantum bit)

Classical information can be transmitted by classical means

Classical information can be transmitted by quantum means

Classical spin-1/2 particles (just two states): \[ \begin{align*}
| \uparrow \rangle & \text{ spin up} \\
| \downarrow \rangle & \text{ spin down}
\end{align*} \]

Quantum information can be transmitted by quantum means Quantum spin-1/2 particles (infinitely many states)

\[ | \psi \rangle = \alpha | \uparrow \rangle + \beta | \downarrow \rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathbb{C}^2 \]

where \( \alpha \) and \( \beta \) complex parameters, \( | \psi \rangle \) denotes the one spin state.

Quantum information \( \equiv \) quantum entanglement (it has no classical counterpart)

\[ | \beta_{00} \rangle = \frac{1}{\sqrt{2}} (| \uparrow \uparrow \rangle + | \downarrow \downarrow \rangle) = \frac{1}{\sqrt{2}} (| \uparrow \rangle \otimes | \uparrow \rangle + | \downarrow \rangle \otimes | \downarrow \rangle) \]

(\( \otimes \) denotes the tensor product)

Entanglement is a physical source. (For entanglement we need two systems)

Classical information : \[ \begin{align*}
\text{Resources : } & \text{ CBIT}(| \uparrow \rangle, | \downarrow \rangle) \\
\text{Operation : } & \text{ classical gates}
\end{align*} \]

Two operations on classical bits

Identity operation : \[ \begin{align*}
| \uparrow \rangle & \rightarrow | \uparrow \rangle \\
| \downarrow \rangle & \rightarrow | \downarrow \rangle
\end{align*} \]

Flip operation : \[ \begin{align*}
| \uparrow \rangle & \rightarrow | \downarrow \rangle \\
| \downarrow \rangle & \rightarrow | \uparrow \rangle
\end{align*} \]
Quantum information:

\[
\begin{cases}
\text{Resources:} & 
\begin{cases}
\text{CBIT}(|\uparrow\rangle, |\downarrow\rangle) \\
\text{QUBIT}(\alpha|\uparrow\rangle + \beta|\downarrow\rangle)
\end{cases} \\
\text{Operation:} & \text{entanglement}
\end{cases}
\]

Infinitely many operations on quantum bits

**HILBERT SPACES**

Hilbert Space \((H)\) is a linear space with a scalar product. 

Linearity: \(|\psi\rangle, |\phi\rangle \in H\) and \(\alpha, \beta \in C \Rightarrow \alpha|\psi\rangle + \beta|\phi\rangle \in H\)

\[
H = C^2, |\psi\rangle = \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) = \alpha \left( \begin{array}{c} 1 \\ 0 \end{array} \right) + \beta \left( \begin{array}{c} 0 \\ 1 \end{array} \right)
\]

\[
|0\rangle = |\uparrow\rangle = \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \\
|1\rangle = |\downarrow\rangle = \left( \begin{array}{c} 0 \\ 1 \end{array} \right)
\]

\(|0\rangle\) and \(|1\rangle\) are the basis for \(H = C^2\).

**Scalar Product** : \((|\phi\rangle, |\psi\rangle) = \langle \phi|\psi\rangle \in C\) where \(|\psi\rangle, |\phi\rangle \in H\)

\[
\forall |\psi\rangle, |\psi_1\rangle, |\psi_2\rangle, |\phi\rangle \in H\) and \(\alpha, \beta \in C\)

Linearity of scalar product on the right:

\[
\langle \phi| (\alpha|\psi_1\rangle + \beta|\psi_2\rangle) = \alpha\langle \phi|\psi_1\rangle + \beta\langle \phi|\psi_2\rangle
\]

Antilinearity of scalar product on the left:

\[
(\alpha\langle \psi_1| + \beta\langle \psi_2|) |\phi\rangle = \overline{\alpha}\langle \psi_1|\phi\rangle + \overline{\beta}\langle \psi_2|\phi\rangle
\]

Positivity of scalar product:

\[
\langle \psi|\psi\rangle = ||\psi||^2 \geq 0 \quad (= 0 \Leftrightarrow |\psi\rangle = 0)
\]

where \(\overline{\alpha}\) and \(\overline{\beta}\) denotes the complex conjugate of the complex numbers \(\alpha\) and \(\beta\).
\[ |\psi\rangle + 0 = |\psi\rangle \, (0 \text{ is the null vector in the two dimensional Hilbert space}) \]

\[ \langle \phi | \psi \rangle = \langle \psi | \phi \rangle \]

\[ |\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \text{ and } |\phi\rangle = \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \Rightarrow \langle \phi | \psi \rangle = (\begin{pmatrix} \gamma \\ \delta \end{pmatrix}) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \overline{\gamma} \alpha + \delta \beta \]

\[ |\phi\rangle = \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \Rightarrow \langle \phi | = (\begin{pmatrix} \gamma \\ \delta \end{pmatrix}) \]

\[ \langle 0 | 1 \rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 = \langle 1 | 0 \rangle \]

\[ |0\rangle \text{ and } |1\rangle \text{ are orthogonal.} \]

\[ \langle 0 | 0 \rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \]

\[ \langle 1 | 1 \rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 \]

\[ |0\rangle \text{ and } |1\rangle \text{ are orthonormal. (orthonormal basis for } C^2, \text{ ONB for } C^2) \]

\[ |\psi\rangle \in H = C^d \Rightarrow |\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \\ ... \\ \psi_d \end{pmatrix} = \psi_1 \begin{pmatrix} 1 \\ 0 \\ ... \\ 0 \end{pmatrix} + \psi_2 \begin{pmatrix} 1 \\ 0 \\ ... \\ 0 \end{pmatrix} + ... + \psi_d \begin{pmatrix} 1 \\ 0 \\ ... \\ 0 \end{pmatrix} \]

Since \( \langle j | k \rangle = \delta_{jk} \) where

\[ |0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, ..., |d\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \]

then \( \{|i\rangle\}_{i=1}^{d} \) is ONB for d dimensional Hilbert space \( C^d \).

Quantum systems described in Hilbert space. \( |\psi\rangle \)’s are the states of the quantum systems.

Projection operator on the \( |\psi\rangle \), \( P_{\psi} = |\psi\rangle \langle \psi| \) where \( ||\psi|| = 1 \)

\[ P_{\psi} |\phi\rangle = |\psi\rangle \langle \psi| |\phi\rangle = \langle \psi | \phi \rangle |\psi\rangle \]

(since \( \langle \psi | \phi \rangle \) is scalar then \( \langle \psi | \phi \rangle \) end the vector \( |\psi\rangle \) can exchange their positions)

\( P_{\psi} \) is a linear operator in \( H \).

\[ P_{\psi}^2 = P_{\psi} P_{\psi} = (|\psi\rangle \langle \psi|) (|\psi\rangle \langle \psi|) = |\psi\rangle \langle \psi| |\psi\rangle \langle \psi| = |\psi\rangle \langle \psi| = P_{\psi} \]
Linear Operators : $A : H \to H, |\psi_1\rangle, |\psi_2\rangle \in H$ and $\alpha, \beta \in \mathbb{C}$

$$A(\alpha|\psi_1\rangle + \beta|\psi_2\rangle) = \alpha A|\psi_1\rangle + \beta A|\psi_2\rangle$$

$A$ is a linear operator in $H = \mathbb{C}^2$ is a $2 \times 2$ matrix.

$$A \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} a_{11}\alpha + a_{12}\beta \\ a_{21}\alpha + a_{22}\beta \end{pmatrix} \in \mathbb{C}^2$$

$P_0 = |0\rangle\langle 0|$ \hspace{1cm} $P_\psi = |\psi\rangle\langle \psi|$ \hspace{1cm} $P_\psi (\alpha|\phi_1\rangle + \beta|\phi_2\rangle) = |\psi\rangle\langle \psi| (\alpha|\phi_1\rangle + \beta|\phi_2\rangle) = |\psi\rangle (\alpha\langle \psi|\phi_1 \rangle + \beta\langle \psi|\phi_2 \rangle)$

$$\begin{align*}
P_0 \begin{pmatrix} \alpha \\ \beta \end{pmatrix} &= |0\rangle\langle 0| \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \alpha|0\rangle \\
P_\psi &= \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \alpha|0\rangle
\end{align*}$$

$P_\psi$ is a linear operator such that $P_\psi^2 = P_\psi$ and $P_\psi^\dagger = P_\psi$ (self adjoint)

Self Adjoint Operator : $\dagger$ takes the adjoint.

Let $A$ be a linear operator on $H = \mathbb{C}^d$ and $|\psi\rangle, |\phi\rangle \in \mathbb{C}^d$

$A \to A^\dagger$ such that $\langle \phi | (A|\psi\rangle) = \langle A^\dagger \phi | \psi \rangle$

$H = \mathbb{C}^2 \Rightarrow A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, |\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathbb{C}^2, |\phi\rangle = \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \in \mathbb{C}^2$

$$\begin{align*}
\langle \phi | (A|\psi\rangle) &= (\gamma \quad \delta) \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = a_{11}\gamma\alpha + a_{12}\gamma\beta + a_{21}\delta\alpha + a_{22}\delta\beta \\
\text{adjoint of } A \text{ is } A^\dagger = (A^T)^* = (A^*)^T \text{ (where } T \text{ denotes the transposition and } * \text{ denotes the complex conjugation operations).}
\end{align*}$$

$$A^* = \begin{pmatrix} a_{11}^* & a_{12}^* \\ a_{21}^* & a_{22}^* \end{pmatrix} \Rightarrow (A^*)^T = \begin{pmatrix} a_{11}^* & a_{12}^* \\ a_{21}^* & a_{22}^* \end{pmatrix} = A^\dagger$$
\[ A^{\dagger}|\phi\rangle = \begin{pmatrix} a_{11}^* & a_{21}^* \\ a_{12}^* & a_{22}^* \end{pmatrix} \begin{pmatrix} \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} a_{11}^* \gamma + a_{21}^* \delta \\ a_{12}^* \gamma + a_{22}^* \delta \end{pmatrix} \]

Bra of \( A^{\dagger}|\phi\rangle = (a_{11} \gamma + a_{21} \delta \ a_{12} \gamma + a_{22} \delta) \)

\[ \langle A^{\dagger}|\psi\rangle = \begin{pmatrix} a_{11} \gamma + a_{21} \delta & a_{12} \gamma + a_{22} \delta \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = a_{11} \gamma \alpha + a_{21} \delta \alpha + a_{12} \gamma \beta + a_{22} \delta \beta \]

\( A \) is self-adjoint iff (if and only if) \( A = A^{\dagger} \)

\( P_\psi \) is self-adjoint, \( P_\psi = |\psi\rangle \langle \psi| \)

\[ P_\psi^{\dagger} = (|\psi\rangle \langle \psi|)^\dagger = |\psi\rangle \langle \psi| \Rightarrow P_\psi = P_\psi^{\dagger} \]

The self-adjoint operators (\( A = A^{\dagger} \)) are the observables of the quantum systems (one can experimentally measure the observables)

*The eigenvalues of \( A \)

\( A|a\rangle = a|a\rangle \): eigenvalue equation. (\( a \): eigenvalue of \( A \), \( |a\rangle \): eigenvector of \( A \))

For self-adjoint operators all eigenvalues are real.

Eigenvalues are the possible outcomes of the measurement of \( A \).

For example:

\[ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma_z^{\dagger} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow \sigma_z = \sigma_z^{\dagger} \]

eigenvalues are +1 and -1.

The outcome of the measurement depends on the state of the system (\( |\psi\rangle \)).

*Spectral representation of \( A = A^{\dagger} \):

\[ A = \sum_i a_i|a_i\rangle \langle a_i| \]

where \( H = C^d \), \( A|a_i\rangle = a_i|a_i\rangle \), \( i = 1, 2, ..., d \)
For example:

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = (+1) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + (-1) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Since

$$|0\rangle\langle 0| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$|1\rangle\langle 1| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

then

$$\sigma_z = (+1) |0\rangle\langle 0| + (-1) |1\rangle\langle 1|$$

Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 \\ -i \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Reduction Postulate:

A is the observable to be measured.

$$A = \sum_{i=1}^{d} a_i |a_i\rangle\langle a_i|$$

$|\psi\rangle$ is the state of the system while measuring $A$. The result of the measuring $A$ on $|\psi\rangle$ is getting the eigenvalues $a_i$ with probabilities:

$$p_{\psi}(a_i) = |\langle \psi | a_i \rangle|\geq 0$$

where $a_i$ and $|a_i\rangle$ are the eigenvalues and the eigenvectors of A ($A|a_i\rangle = a_i|a_i\rangle$).

After the measurement the quantum system will be in the state $|a_i\rangle$. In quantum mechanics, measurement destroys the state of the system.

$$p_{\psi}(a_i) = |\langle \psi | a_i \rangle|^2 \geq 0 \quad \text{and} \quad A = \sum_{i=1}^{d} |\langle \psi | a_i \rangle|^2 = 1$$

$$A = A^\dagger = \sum_{i=1}^{d} a_i |a_i\rangle\langle a_i|, a_i \in \mathbb{R}$$

$P_{a_i} = |a_i\rangle\langle a_i|$ are orthogonal.
\[ A|a_i\rangle = a_i|a_i\rangle, \langle a_i|a_j\rangle = \delta_{ij} \] (For self-adjoint operators, the eigenvalues which correspond to the different eigenvalues are orthogonal)

\[ P_a P_a = \langle a_i \langle a_i \rangle \langle a_j \langle a_j \rangle \rangle = \langle a_i \langle a_i \rangle \langle a_j \rangle \rangle = \delta_{ij} \langle a_i \rangle \langle a_j \rangle = \langle a_i \rangle \langle a_i \rangle = P_a \]

Eigenvectors of \( A = A^\dagger \) are orthonormal and form a basis in \( C^d \).

|\psi\rangle \in C^d, |\psi\rangle = \sum_{i=1}^{d} c_i |a_i\rangle = \sum_{i=1}^{d} \langle a_i | \psi \rangle |a_i\rangle

where \( |a_i\rangle \) are the eigenvectors of \( A = A^\dagger \).

\[ \langle a_j | \psi \rangle = \langle a_j | \left( \sum_{i=1}^{d} c_i |a_i\rangle \right) = \sum_{i=1}^{d} c_i \langle a_j | a_i\rangle = \sum_{i=1}^{d} c_i \delta_{ji} = c_j \]

\[ |\psi\rangle = \sum_{i=1}^{d} \langle a_i | \psi \rangle |a_i\rangle = \sum_{i=1}^{d} \langle a_i \rangle \langle a_i | \psi \rangle \]

(since \( \langle a_i | \psi \rangle \) is a scalar it can exchange the position in the product with the vector \( |a_i\rangle \))

\[ \Rightarrow |\psi\rangle = \sum_{i=1}^{d} P_a |\psi\rangle \]

\[ \sum_{i=1}^{d} p_{\psi} (a_i) = \sum_{i=1}^{d} \langle \psi | a_i \rangle^2 = \sum_{i=1}^{d} \langle \psi | a_i \rangle \langle a_i | \psi \rangle = \sum_{i=1}^{d} \langle \psi | P_a | \psi \rangle = \langle \psi | \sum_{i=1}^{d} P_a |\psi\rangle \]

(since the summation is linear)

\[ \Rightarrow \sum_{i=1}^{d} p_{\psi} (a_i) = \langle \psi | \psi \rangle = 1 \]

(since \( |\psi\rangle \) is normalized)

\textit{Cauchy - Schwartz Inequality :} \( \forall |\psi\rangle, |\phi\rangle \in H \)

\[ |\langle \phi | \psi \rangle| \leq \| \phi \| \| \psi \| = \sqrt{\langle \phi | \phi \rangle} \sqrt{\langle \psi | \psi \rangle} \]

Proof: Let \( |\chi\rangle = |\phi\rangle + \mu |\psi\rangle \) and from the positivity of the scalar product \( \langle \chi | \chi \rangle \geq 0 \)

\[ \langle \chi | \chi \rangle = \langle \phi | \phi \rangle + \mu \langle \phi | \psi \rangle + \mu \langle \phi | \psi \rangle + \mu \langle \psi | \psi \rangle \]

Let

\[ \langle \phi | \psi \rangle = |\langle \phi | \psi \rangle| e^{i\theta} \] and \( \langle \psi | \phi \rangle = |\langle \phi | \psi \rangle| e^{-i\theta} \)

(from the property of the scalar product \( |\langle \phi | \psi \rangle| = |\langle \psi | \phi \rangle| \)) where \( \theta = \theta (\psi, \phi) \) and let \( \mu = xe^{-i\theta} \) then,

\[ \langle \chi | \chi \rangle = \langle \phi | \phi \rangle + |\langle \phi | \psi \rangle| (\mu e^{i\theta} + \mu e^{-i\theta}) + |\mu|^2 \langle \psi | \psi \rangle \]

\[ = \langle \phi | \phi \rangle + 2x \langle \phi | \psi \rangle x^2 \langle \psi | \psi \rangle \]
This is the parabola \( f(x) \), since \( \langle \chi | \chi \rangle \geq 0 \) then parabola must be positive (i.e. has no root)

\[
f(x) = 0 \Rightarrow x_{\pm} = -\|\langle \phi | \psi \rangle\| \pm \sqrt{\|\langle \phi | \psi \rangle\|^2 - \|\phi\|^2 \|\psi\|^2}
\]
since \( f(x) \) must has no root then inside of the square root in the last expression must be zero or less than zero.

\[
|\langle \phi | \psi \rangle| \leq \|\phi\| \|\psi\|
\]

If the vector is not normalized it can be normalized, i.e.

\[
\langle \tilde{\psi} | \tilde{\psi} \rangle \neq 1 \Rightarrow |\psi\rangle = \frac{\tilde{\psi}}{\|\tilde{\psi}\|} \quad \text{then} \quad \langle \psi | \psi \rangle = 1
\]

\[
|\psi\rangle = e^{i\phi}|\psi\rangle \quad \text{where} \quad e^{i\phi} \quad \text{is the phase factor. Let} \quad |\psi\rangle = e^{ix}|\psi\rangle \quad \text{and} \quad |a_i\rangle = e^{iy}|a_i\rangle
\]

then are the probabilities change under this transformations?

\[
p_{\psi}(a_i) = \frac{\langle \psi | a_i \rangle^2}{\|a_i\|^2} = \frac{|\langle \psi | e^{-ix}e^{iy}a_i \rangle|^2}{\|a_i\|^2} = e^{i(y-x)}\langle \psi | a_i \rangle^2 = \frac{|\langle \psi | a_i \rangle|^2}{\|a_i\|^2} = p_{\psi}(a_i)
\]

The probabilities don’t change under the phase transformation.

Let \( A \) be \( d \times d \) matrix.

\[
A = \begin{pmatrix}
da_11 & a_{12} & \ldots & a_{1d} \\
da_{21} & a_{22} & \ldots & a_{2d} \\
\vdots & \vdots & \ddots & \vdots \\
da_{d1} & a_{d2} & \ldots & a_{dd}
\end{pmatrix}
\]

\( Tr(A) \equiv \text{sum of the diagonal elements of } A \).

\[
Tr(A) = a_{11} + a_{22} + ... + a_{dd} = \sum_{i=1}^{d} a_{ii}
\]

\( (A, B) \rightarrow \langle \langle A, B \rangle \rangle = Tr \left( A^\dagger B \right) \) From the Cauchy - Schwartz inequality,

\[
Tr \left( A^\dagger B \right) \leq \sqrt{Tr (A^\dagger A)} \sqrt{Tr (B^\dagger B)}
\]

Prove that \( \langle \psi | \phi \rangle = \overline{\langle \phi | \psi \rangle} \quad \text{(suggestion : with using} \quad \langle \phi + \psi | \phi + \psi \rangle, \langle \phi - \psi | \phi - \psi \rangle, \langle \phi + i\psi | \phi + i\psi \rangle, \langle \phi - i\psi | \phi - i\psi \rangle \quad \text{construct the} \quad \langle \phi | \psi \rangle \quad \text{and} \quad \langle \psi | \phi \rangle) \)

From the linearity and anti linearity properties of the scalar product,

\[
\langle \phi + \psi | \phi + \psi \rangle = \langle \phi | \phi \rangle + \langle \phi | \psi \rangle + \langle \psi | \phi \rangle + \langle \psi | \psi \rangle \equiv F_1
\]

\[
\langle \phi - \psi | \phi - \psi \rangle = \langle \phi | \phi \rangle - \langle \phi | \psi \rangle - \langle \psi | \phi \rangle + \langle \psi | \psi \rangle \equiv F_2
\]

\[
\langle \phi + i\psi | \phi + i\psi \rangle = \langle \phi | \phi \rangle + i\langle \phi | \psi \rangle - i\langle \psi | \phi \rangle + \langle \psi | \psi \rangle \equiv F_3
\]

\[
\langle \phi - i\psi | \phi - i\psi \rangle = \langle \phi | \phi \rangle - i\langle \phi | \psi \rangle + i\langle \psi | \phi \rangle + \langle \psi | \psi \rangle \equiv F_4
\]
\[ F_1 - F_2 = \langle \phi | \psi \rangle + 2 \langle \psi | \phi \rangle \]
\[ F_3 - F_4 = 2i \langle \phi | \psi \rangle - 2i \langle \psi | \phi \rangle \]
\[ \Rightarrow -i (F_3 - F_4) = 2 \langle \phi | \psi \rangle - 2 \langle \psi | \phi \rangle \]
\[ (F_1 - F_2) - i (F_3 - F_4) = 4 \langle \phi | \psi \rangle \Rightarrow \phi | \psi \rangle = \frac{F_1 - F_2}{4} - i \frac{F_3 - F_4}{4} \]
\[ (F_1 - F_2) + i (F_3 - F_4) = 4 \langle \psi | \phi \rangle \Rightarrow \psi | \phi \rangle = \frac{F_1 - F_2}{4} + i \frac{F_3 - F_4}{4} \]

Since \( \langle . | . \rangle \geq 0 \) then \( F_i \) (\( i = 1, 2, 3, 4 \)) are real then \( \langle \psi | \phi \rangle = \overline{\langle \phi | \psi \rangle} \).

\{ |e_i \rangle \} ONB for \( C^d \), \( \langle e_i | e_j \rangle = \delta_{ij} \), \( P_{e_i} = |e_i \rangle \langle e_i | \), \( \sum_{i=1}^{d} P_{e_i} = 1 \)

\[ |\langle \phi | \psi \rangle| = |\langle \phi | 1 | \psi \rangle| = \left| \langle \phi \right| \sum_{i=1}^{d} P_{e_i} |\psi \rangle \right| = \left| \sum_{i=1}^{d} \langle \phi | P_{e_i} |\psi \rangle \right| \]

(since the scalar product is linear)

\[ \Rightarrow |\langle \phi | \psi \rangle| = \left| \sum_{i=1}^{d} \langle \phi | e_i \rangle \langle e_i | \psi \rangle \right| \]

\[ |\phi \rangle = \sum_{i=1}^{d} \langle e_i | \phi \rangle |e_i \rangle \text{ and } |\psi \rangle = \sum_{i=1}^{d} \langle e_i | \psi \rangle |e_i \rangle \text{ then,} \]

\[ \Rightarrow |\langle \phi | \psi \rangle| = \left| \sum_{i=1}^{d} \phi_i^* \psi_i \right| \]

Every linear operator can be represented by matrices.

\[ A : C^d \to C^d \), \( |\psi \rangle \in C^d \mid \psi \rangle = \begin{pmatrix} \psi_1 \\
\psi_2 \\
\vdots \\
\psi_d \end{pmatrix} = \sum_{i=1}^{d} \psi_i |i \rangle \]

where \( |i \rangle = \begin{pmatrix} 0 \\
\vdots \\
1 \\
\vdots \end{pmatrix} \) (all elements are 0 except the \( i \) th row (it is 1))

\[ |\psi \rangle \xrightarrow{A} A|\psi \rangle \in C^d \), \( \{ |i \rangle \} \) ONB for \( H = C^d \)

\[ A|\psi \rangle = A \left( \sum_{i=1}^{d} \psi_i |i \rangle \right) = \sum_{i=1}^{d} \psi_i A|i \rangle \]

(since \( A \) is linear)
If we know the action of the $A$ on the basis vectors, then we can calculate the action of the $A$ on all vectors in $C^d$.

$$A|i⟩ = \sum_{j=1}^{d} c_j^i |j⟩, c_j^i = ⟨j|A|i⟩$$

(matrix element of $A$ on the $jth$ row and $ith$ column)

$$\sum_{i=1}^{d} \psi_i A|i⟩ = \sum_{i=1}^{d} \psi_i \sum_{j=1}^{d} ⟨j|A|i⟩ |j⟩ = \sum_{j=1}^{d} \left( \sum_{i=1}^{d} ⟨j|A|i⟩ \psi_i \right) |j⟩$$

$$A = \begin{pmatrix}
⟨1|A|1⟩ & ⟨1|A|2⟩ & \cdots & ⟨1|A|d⟩ \\
⟨2|A|1⟩ & ⟨2|A|2⟩ & \cdots & ⟨2|A|d⟩ \\
\vdots & \vdots & \ddots & \vdots \\
⟨d|A|1⟩ & ⟨d|A|2⟩ & \cdots & ⟨j|A|d⟩
\end{pmatrix}$$

$$|ψ'⟩ = A|ψ⟩$$

$$\begin{pmatrix}
ψ'_1 \\
ψ'_2 \\
\vdots \\
ψ'_d
\end{pmatrix} = \begin{pmatrix}
⟨1|A|1⟩ & ⟨1|A|2⟩ & \cdots & ⟨1|A|d⟩ \\
⟨2|A|1⟩ & ⟨2|A|2⟩ & \cdots & ⟨2|A|d⟩ \\
\vdots & \vdots & \ddots & \vdots \\
⟨d|A|1⟩ & ⟨d|A|2⟩ & \cdots & ⟨j|A|d⟩
\end{pmatrix} \begin{pmatrix}
ψ_1 \\
ψ_2 \\
\vdots \\
ψ_d
\end{pmatrix}$$

$P_ψ = P_ψ^\dagger = P_ψ^2$ projection operator on the $|ψ⟩ \in H = C^d$

$$P_ψ = |ψ⟩⟨ψ|$$

and $\{|i⟩\}_{i=1}^{d}$ ONB for $H = C^d$

$$|ψ⟩ = \sum_{i=1}^{d} ψ_i |i⟩$$

$$⇒ P_ψ = |ψ⟩⟨ψ| = \sum_{i,j=1}^{d} ψ_i ψ_j ⟨i|j⟩ = \begin{pmatrix}
|ψ_1|^2 & ψ_1 ψ_2^* & \cdots & ψ_1 ψ_d^* \\
ψ_2 ψ_1^* & |ψ_2|^2 & \cdots & ψ_2 ψ_d^* \\
\vdots & \vdots & \ddots & \vdots \\
ψ_d ψ_1^* & ψ_d ψ_2^* & \cdots & |ψ_d|^2
\end{pmatrix}$$

$$P_ψ^\dagger = [(|ψ⟩⟨ψ|)^*]^T = \begin{pmatrix}
|ψ_1|^2 & ψ_1^* ψ_2 & \cdots & ψ_1^* ψ_d \\
ψ_2^* ψ_1 & |ψ_2|^2 & \cdots & ψ_2^* ψ_d \\
\vdots & \vdots & \ddots & \vdots \\
ψ_d^* ψ_1 & ψ_d^* ψ_2 & \cdots & |ψ_d|^2
\end{pmatrix}$$

Thus, $P_ψ = P_ψ^\dagger$.

In any case $\{|e_i⟩\}_{i=1}^{d}$ ONB for $H = C^d$ ($⟨e_i|e_j⟩ = δ_{ij}$)

$$|ψ⟩ = \sum_{i=1}^{d} ⟨e_i|ψ⟩ |e_i⟩$$
\[ \Rightarrow A|\psi \rangle = A \left( \sum_{i=1}^{d} \langle e_i | \psi \rangle e_i \right) = \sum_{i=1}^{d} \langle e_i | \psi \rangle A|e_i \rangle \quad \text{(since A is linear)} \]

\[ = \sum_{i=1}^{d} \langle e_i | \psi \rangle \sum_{j=1}^{d} \langle e_j | A|e_i \rangle |e_j \rangle \]

\[ = \sum_{j=1}^{d} \left( \sum_{i=1}^{d} \langle e_j | A|e_i \rangle \langle e_i | \psi \rangle \right) |e_j \rangle \]

\[ A = [\langle e_i | A|e_j \rangle]_{d \times d} \]

\[ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

\[ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = (+1) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]

\[ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = (-1) \begin{pmatrix} 1 \\ -1 \end{pmatrix} \]

The eigenvalues of \( \sigma_x \) are +1, −1 and their corresponding eigenvectors are \( \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) (normalized)

\[ |+x \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]

\[ |-x \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \]

with calculating the matrix elements of \( \sigma_x \) in the ONB \( \{|+x\rangle, |-x\rangle\} \)

\[ \sigma_x = \begin{pmatrix} \langle +x | \sigma_x | +x \rangle & \langle +x | \sigma_x | -x \rangle \\ \langle -x | \sigma_x | +x \rangle & \langle -x | \sigma_x | -x \rangle \end{pmatrix} \]

\[ \langle +x | \sigma_x | +x \rangle = \langle +x | +x \rangle = 1 \]

\[ \langle +x | \sigma_x | -x \rangle = -\langle +x | -x \rangle = 0 \]

\[ \langle -x | \sigma_x | +x \rangle = \langle -x | +x \rangle = 0 \]

\[ \langle -x | \sigma_x | -x \rangle = -\langle -x | -x \rangle = -1 \]

\[ \sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

(in the basis \( \{|+x\rangle, |-x\rangle\} \))

While the basis changing, the properties of the \( \sigma_i \)’s

\[ Tr (\sigma_i) = 0 \]

\[ Det (\sigma_i) = -1 \]

\[ (\sigma_i)^2 = 1 \]

don’t change. Changing representation doesn’t change the physical mean.

\[ \{ |i \rangle \}_{i=1}^{d} \xrightarrow{U} \{ |e_i \rangle \}_{i=1}^{d} \Rightarrow \sigma_i = U \sigma_i U^\dagger \]
(U : unitary transformation)
\[ \sigma_x \sigma_y = i \sigma_z \xrightarrow{U} U \sigma_x U^\dagger U \sigma_y U^\dagger = i U \sigma_z U^\dagger \]
\[ \Rightarrow U \sigma_x \sigma_y U^\dagger = U i \sigma_z U^\dagger \]
(since U is unitary then \(UU^\dagger = 1\))
\[ \sigma_x \sigma_y = i \sigma_z \]
(changing basis don’t change the form of equality \(\sigma_x \sigma_y = i \sigma_z\))

Summary of the previous lecture
\[ A = A^\dagger = \sum_{i=1}^d a_i |a_i > < a_i| \quad A|a_i >= a_i|a_i > \]
\[ p_\psi (a_i) = | < \psi | a_i > |^2 \quad \sum_{i=1}^d p_\psi (a_i) = 1 \]
\[ P_{a_i} P_{a_j} = \delta_{ij} \quad \sum_{i=1}^d P_{a_i} = 1 \]
\[ p_\psi (a_i) = | < \psi | a_i > |^2 = < \psi | a_i > < a_i | \psi > = < \psi | P_{a_i} | \psi > = E_\psi (P_{a_i}) \]
\[ E_\psi (A) : \text{Expectation value of } A \text{ with respect to } \psi \]
\[ E_\psi (A) = \sum_{i=1}^d a_i p_\psi (a_i) = \sum_{i=1}^d a_i < \psi | P_{a_i} | \psi > \]
\[ = < \psi | \left( \sum_{i=1}^d a_i P_{a_i} \right) | \psi > = < \psi | A | \psi > \]

Definition : The self-adjoint matrices (\(A = A^\dagger\)) in the full \(d \times d\) matrix algebra \((M_d(C))\). \(M_d(C)\) are the physical observables.

Definition : Every \(|\psi > \in C^d\), \(||\psi || = 1\) is a pure state on \(M_d(C)\) which fixes the expectation values of all
\[ A = A^\dagger \in M_d(C) . E_\psi (A) = < \psi | A | \psi > \]

Definition : The trace of a matrix \(B \in M_d(C)\) is defined by
\[ Tr (B) = \sum_{i=1}^d < \psi_i | B | \psi_i > \quad \text{where } | \{ |\psi_i > \}^d_{i=1} \text{ is any ONB in } C^d \]

- The trace is basis independent.
- \(Tr (B) = \sum_{i=1}^d B_{ii}, \quad B = [B_{ij}]_{d \times d}, \quad B = [B_{ij}]^d_{i,j=1} \)
• $Tr(AB) = Tr(BA)$

In classical mechanics $fg = gf$, $f, g \in C_{\infty}(x)$ where $x$ is a phase space coordinate $(q, p)$.

In quantum mechanics $[A, B] = AB - BA \neq 0$ (in general), $A, B \in M_d(C)$.

Classical mechanics; $q, p$; $\{q, p\} = 1$ (Poisson Bracket)

Quantum mechanics; $\hat{q}, \hat{p}$; $[\hat{q}, \hat{p}] = i\hbar$ (Commutator)

Since $Tr(AB) = Tr(BA)$ then $Tr([A, B]) = 0$

$$E_\psi(A) = \langle \psi | A | \psi \rangle = Tr(\langle \psi > < \psi | A | \psi \rangle) = Tr(P_\psi A)$$

Proof:

$$Tr(P_\psi A) = \sum_{i=1}^{d} \langle \phi_i | P_\psi A | \phi_i \rangle; \{ | \phi_i \rangle \}_{i=1}^d \text{ is ONB in } C^d$$

Since trace is basis independent one can set $| \phi_1 > = | \psi >$, complementary space is $C^{d-1}$, $\{ | \phi_2 >, | \phi_3 >, ..., | \phi_d > \}$, then $\{ | \psi >, | \phi_2 >, | \phi_3 >, ..., | \phi_d > \}$ is ONB in $C^d$.

$$Tr(P_\psi A) = \sum_{i=1}^{d} \langle \phi_i | P_\psi A | \phi_i \rangle = \langle \psi | P_\psi A | \psi \rangle + \sum_{i=2}^{d} \langle \phi_i | P_\psi A | \phi_i \rangle$$

$$= \langle \psi | \psi \rangle < \psi | A | \psi \rangle + \sum_{i=2}^{d} \langle \phi_i | \psi \rangle < \phi_i | A | \phi_i \rangle$$

$$\Rightarrow Tr(P_\psi A) = \langle \psi | A | \psi \rangle$$

(From the orthogonality) This was for the pure state.

**Definition Mixed States**: (to be compared with the pure states $P_\psi$) are linear convex combinations of pure states. (density matrix, mixed states)

$$P_{\psi_i} \quad (i = 1, 2, ..., m), \quad 0 \leq \lambda_i \leq 1, \sum_{i=1}^{m} \lambda_i = 1, \sum_{i=1}^{m} \lambda_i P_{\psi_i} = \rho$$

where $\lambda_i$ are the weights

Since $P_{\psi_i}$ are the pre states then the mixed state is the mixture of the pre states. (e.g., two incoming beam. One is consist of spin up in $z$ direction (40%), the other is consist of spin up in direction $x$ (60%))

$P_\psi$ (pure state): perfect knowledge, Von Neumann entropy is 0.
\(\rho\) (mixed state): partial knowledge, Von Neumann entropy is > 0.

\[
\rho = \sum_{i=1}^{m} \lambda_i P_{\psi_i}, \rho = \rho^\dagger
\]

(since \(\lambda_i \in R\) and \(P_{\psi_i} = P_{\psi_i}^\dagger\))

\[
\rho = \rho^\dagger = \sum_{j=1}^{d} \rho_j |\rho_j><\rho_j|
\]

while \(\lambda_i\)'s are weights, the eigenvalues of \(\rho\) are \(s(\rho_j)\) the probabilities.

The eigenvalues of any density matrix \(\rho\) are the probabilities.

\[
\rho_j \geq 0 \text{ and } \sum_{j=1}^{d} \rho_j = 1, \sum_{j=1}^{d} \rho_j |\rho_j><\rho_j| \text{ in the basis } \{|\rho_j>\}_{j=1}^{d}
\]

\[
\rho = \begin{pmatrix}
\rho_1 & 0 & 0 & 0 \\
0 & \rho_2 & 0 & 0 \\
... & ... & ... & ... \\
0 & 0 & 0 & \rho_d
\end{pmatrix}
\]

\[
Tr(\rho) = Tr \left( \sum_{j=1}^{d} \rho_j |\rho_j><\rho_j| \right)
\]

\[
= \sum_{j=1}^{d} \rho_j Tr(|\rho_j><\rho_j|) = \sum_{j=1}^{d} \rho_j
\]

(Because of \(Tr(|\rho_j><\rho_j|) = |\rho_j><\rho_j|= 1\))

We have proved that \(\sum_{i=1}^{d} \rho_i = Tr(\rho)\) and since \(\rho = \sum_{i=1}^{m} \lambda_i P_{\psi_i}\) then

\[
Tr(\rho) = Tr \left( \sum_{i=1}^{m} \lambda_i P_{\psi_i} \right) = \sum_{i=1}^{m} \lambda_i Tr(P_{\psi_i}) = 1
\]

(since \(Tr(P_{\psi_i}) = 1\))

For the pure state all \(\lambda_i\)'s are 0 except \(i = j\) and \(\lambda_j = 1\). (only one of the \(\lambda_i\) is one all others are zero)

In general any density matrix \(Tr(\rho) = 1\) (as seen above)

Pure States \(\equiv\) Projections \(P_\psi = |\psi><\psi|, \|\psi\| = 1\)
Mixed State \( \equiv \sum_{i=1}^{m} \lambda_i P_{\psi_i}, \) \( \sum_{i=1}^{m} \lambda_i = 1 \) and \( 0 \leq \lambda_i \leq 1 \) (density matrices) \( m \) depends on the preparation of the mixture.

\[ |\psi> \rightarrow E_\psi (A) = <\psi|A|\psi> = Tr (P_{\psi} A) \]

then there is one-to-one correspondence between \( |\psi> \rightarrow P_{\psi} \)

\[ \rho = \sum_{i=1}^{m} \lambda_i P_{\psi_i} E_\rho = \sum_{i=1}^{m} \lambda_i E_{\psi_i} (A) \]

Suppose that \( \rho = \frac{2}{5} |\uparrow_z><\uparrow_z| + \frac{3}{5} |\uparrow_x><\downarrow_x| \)

\( \rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \) is this density matrix unique? (NO!)

With same density matrix there are many physical states which represented by the same density matrix. When one gives the density matrix \( \rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \) (\( Tr (\rho) = 1 \) and all eigenvalues are positive) then \( \rho^2 = \rho \) iff (if and only if) \( \rho = |\psi><\psi| \) with \( \|\psi\| = 1 \) (i.e pure state)

**Definition (positivity of a linear operator) :** Any \( A \in M_d (C) \) \( (A = A^\dagger) \) is positive if \( \forall |\psi> \in C^d, \; <\psi|A|\psi> \geq 0 \)

\[ A \geq 0 \Rightarrow A = \sum_{i=1}^{d} a_i |a_i><a_i| \; \text{with} \; a_i \geq 0 \; \text{(prove?)} \]

\[ \rho = \sum_{i=1}^{m} \lambda_i P_{\psi_i}, \sum_{i=1}^{m} \lambda_i = 1 \; \text{and} \; 0 \leq \lambda_i \leq 1 \]

(generic definition of density matrix)

\[ <\psi|\rho|\psi> = \sum_{i=1}^{m} \lambda_i <\psi|P_{\psi_i}|\psi> = \sum_{i=1}^{m} \lambda_i <\psi|\psi_i>^2 \geq 0 \]

Quantum states can be identified with all possible density matrices.

Density matrices are positive linear operators of trace one.

Pure states are those density matrices s.t. (such that) \( \rho^2 = \rho. \)

Density matrices describe physical mixtures; infinitely many physical mixtures can be described by same \( \rho \) unless \( \rho \) is a pre state. (many to one correspondence)

Example :
\( \tau = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) (tracial state)

\[
\begin{align*}
\tau &= \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{2} |\uparrow_z><\uparrow_z| + \frac{1}{2} |\downarrow_z><\downarrow_z| \\
\end{align*}
\]

50% spin up, 50% spin down along \( z \) mixture.

\( \sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad |\uparrow_x> = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |\downarrow_x> = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \)

\[
\begin{align*}
\tau &= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2} |\uparrow_x><\uparrow_x| + \frac{1}{2} |\downarrow_x><\downarrow_x| \\
\end{align*}
\]

50% spin up, 50% spin down along \( x \) mixture.

Thus, the interpretation of the state which is described by \( \tau \) is not unique!!!

\( \tau \): unpolarized beam (50% of up, 50% of down along some axis)

Many physical mixtures can be described by the same density matrix (\( \rho \)).

**VON NEUMANN ENTROPY** (compared with the Shannon Entropy)

After now \( \rho \) denotes the density matrix.

\[
\rho = \sum_{i=1}^{m} \lambda_i \psi_i, \quad \sum_{i=1}^{m} \lambda_i = 1 \quad \text{and} \quad 0 \leq \lambda_i \leq 1
\]

\[
S(\rho) = -Tr(\rho \ln \rho)
\]

where \( \rho = \sum_{i=1}^{d} \rho_i |\rho_i><\rho_i| \quad \text{and} \quad \rho |\rho_i>=\rho_i|\rho_i>
\]

\[
S(\rho) = -\sum_{i=1}^{d} \rho_i \ln \rho_i = -\sum_{i=1}^{d} \rho_i \ln \rho_i
\]

\[
f(g) = \sum_{k=0}^{\infty} c_k \rho^k \Rightarrow f(g) |\rho_i> = \left( \sum_{k=0}^{\infty} c_k \rho^k \right) |\rho_i>
\]

\[
\rho^2 |\rho_i> = \rho \rho |\rho_i> = \rho \rho_i |\rho_i> = \rho_i \rho |\rho_i> = \rho_i^2 |\rho_i>
\]

\[
\Rightarrow \left( \sum_{k=0}^{\infty} c_k \rho^k \right) |\rho_i> = \sum_{k=0}^{\infty} c_k \rho_i^k |\rho_i> = f(g_i) |\rho_i>
\]
\( \eta(x) = \begin{cases} -x \ln x & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \Rightarrow S(\rho) = -\text{Tr}(\rho \ln \rho) = -\text{Tr}(\eta(\rho)) \)

Pure state:
\[
\rho = |\psi><\psi| \\
\rho^2 = (|\psi><\psi|)(|\psi><\psi|) = |\psi><\psi|\psi><\psi| = |\psi><\psi| = \rho
\]

\[
P_\psi|\psi > = +1|\psi > \\
P_\psi|\phi > = 0|\phi >
\]

\( \Rightarrow S(\rho) = 0 \)
(since \(<\phi|\psi > = 0\))

\[
S(\rho) = -\text{Tr}(\rho \ln \rho) = -\sum_{i=1}^{d} \rho_i \ln \rho_i = 1 \ln 1 + \sum_{i=1}^{0} 0 \ln 0 = 0
\]
(for pure states)

Entropy is measure of information gained after the measurement or measure of ignorance before the measurement.

For mixed state \( S(\rho) > 0 \)

**SHANNON ENTROPY**

Classical source of symbols, each click of time emerge a symbol \((a_1, a_2, ..., a_m)\) with definite probability \((p(a_1), p(a_2), ..., p(a_m))\). All probabilities are positive and
\[
\sum_{i=1}^{m} p(a_i) = 1
\]

*Binary source*: It emerge one of the two symbols each click of time, 0 or 1 with probability \( p(0) \) and \( p(1) \).

The statistical of a classical source \((S_{cl})\) is described by the Shannon Entropy.

Discrete probability distribution : \( \prod = \{p(a_i)\} \)

\[
H(\prod) = -\sum_{i=1}^{d} p(a_i) \ln p(a_i)
\]

In binary source if \( p(0) = \frac{1}{2} \) and \( p(1) = \frac{1}{2} \) then \( H(\prod) = \ln 2 \)
\[ H \left( \Pi \right) \rightarrow S \left( \rho \right) = -\sum_{i=1}^{d} \rho_i \ln \rho_i, \quad S \left( \rho \right) = H \left( \Pi \rho \right) \]

where \( \Pi = \{ \rho_i \}_{i=1}^{d} \)

The quantum system can be regarded as source which emerges states with definite weights. \((P_{\psi_1}, P_{\psi_2}, ..., P_{\psi_m})\) with weights \((\lambda_1, \lambda_2, ..., \lambda_m)\) and \(\rho = \sum_{i=1}^{m} \lambda_i P_{\psi_i}\).

Classical and quantum sources are parallel. In quantum sources Von Neumann entropy and pure states are correspond to Shannon entropy and emitted symbols in classical sources.

**TWO QUANTUM SYSTEMS**

Each one described by Hilbert space \(C^d\), (e.g. two spin 1/2 particle \(S_1 + S_2\), each one described in \(C^2\). The Hilbert space for the total system is the tensor product of the two space \(C^2 \otimes C^2\)).

In general, if the system \(S_1\) is described in Hilbert space \(C^d\) and the system \(S_2\) is described in Hilbert space \(C^d\) then the compound system \(S_1 + S_2\) is described by the Hilbert space \(C^d \otimes C^d\). \(C^d \otimes C^d\) is also linear space.

Two system with two states:

\[
\begin{align*}
  x = 0,1, |x >, & \quad |0 >, |1 > \in C^2 \\
  y = 0,1, |y >, & \quad |0 >, |1 > \in C^2 \\
\end{align*}
\]

\[
\Rightarrow \sum_{i,j} c_{ij} |x_i > \otimes |y_j > \in C^2 \otimes C^2
\]

\(\alpha |x_1 > \otimes |y_1 > + \beta |x_2 > \otimes |y_2 >\) is a vector in \(C^2 \otimes C^2\). Since \(C^2 \otimes C^2\) is a Hilbert space we have to describe the inner product in it.

\[
ket \rightarrow \text{corresponding bra} \\
|x > \otimes |y > \rightarrow < x| \otimes < y|
\]

\[
( < x_1| \otimes < y_1|)(|x_2 > \otimes |y_2 >) = < x_1|x_2 < y_1|y_2 >
\]

where \(|x_i >\) is first system’s states and \(|y_i >\) is the second system’s states.

Bell States: \((|\beta_{00} >, |\beta_{01} >, |\beta_{10} >, |\beta_{11} >)\)

\[
|\beta_{00} > = \frac{1}{\sqrt{2}}(|0 > \otimes |0 > + |1 > \otimes |1 >), \quad |0 > = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1 > = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]

\(|0 > \otimes |0 >\) and \(|1 > \otimes |1 >\) are the product states. But \(|\beta_{00} >\) is not a product state (which is the linear combination of two product states). \(|\beta_{00} >\) is a entangled
\[ |\beta_{00} > = \frac{1}{\sqrt{2}} (|0 > \otimes |0 > + |1 > \otimes |1 >) = \frac{1}{\sqrt{2}} (|00 > + |11 >) \]

\[ < \beta_{00} | = \frac{1}{\sqrt{2}} (|0 > \otimes |0 > + |1 > \otimes |1 >) = \frac{1}{\sqrt{2}} (|00 > + |11 >) \]

\[ |\beta_{00} < < \beta_{00} | = \frac{1}{\sqrt{2}} (|00 > + |11 >) \frac{1}{\sqrt{2}} (|00 > + |11 >) \]

\[ = \frac{1}{2} (|00 > \otimes |00 > + |00 > \otimes |11 > + |11 > \otimes |00 > + |11 > \otimes |11 >) \]

\[ = \frac{1}{2} (|00 > \otimes |00 > + |00 > \otimes |11 > + |11 > \otimes |00 > + |11 > \otimes |11 >) \]

\[ |\psi_1 >, |\psi_2 > \in C^d \]

\[ |\phi_1 >, |\phi_2 > \in C^d \]

\[ \Rightarrow (|\psi_1 > \otimes |\phi_1 >) (< \psi_2 | < \phi_2 >) = |\psi_1 > < \psi_2 | \otimes |\phi_1 > < \phi_2 | \]

statistical interpretation of term \( \frac{1}{2} (|00 > \otimes |00 > + |00 > \otimes |11 > + |11 > \otimes |00 > + |11 > \otimes |11 >) \):

50% spin up in z direction, 50% spin down in z direction. (The term \( |i > < i| \otimes |i > < i| \) projects the first system on the state \( |i > \) and projects the second system on the \( |i > \) where \( i = 0, 1 \))

But the interference term \( \frac{1}{2} (|01 > \otimes |01 > + |10 > \otimes |10 >) \) prevents the statistical interpretation (The term \( |i > < j| \otimes |i > < j| \) is not projection (because it is not self-adjoint) where \( i, j = 0, 1 \) and \( i \neq j \))

Interference terms prevents attributions individual properties in any sense, even statistical.

\[ |0 > < 0| \otimes |0 > < 0| = |00 > < 00| \]

\[ |0 > < 1| \otimes |0 > < 1| = |00 > < 11| \]

\[ |1 > < 0| \otimes |1 > < 0| = |11 > < 00| \]

\[ |1 > < 1| \otimes |1 > < 1| = |11 > < 11| \]

But beside these, \( |\beta_{00} > \) is a projection operator in \( C^d \). It can be represented by \( 4 \times 4 \) matrix.

\[ |\beta_{00} > < \beta_{00} | = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

\[ = \rho \]

\( \rho = \rho^t, Tr(\rho) = 1, \rho^2 = \rho \) then it is a projector.

Summary:

\[ S_1 + S_2, H = C^2 \otimes C^2, |\psi > = |\psi_1 > \otimes |\psi_2 > : \text{product state.} \]
Entangled state is linear combination of product states.

\[ |\psi> = \sum_{i,j} c_{i,j} |\psi_i> \otimes |\psi_j> \text{, } |\psi_i> \in H_1 = C^d, |\psi_j> \in H_2 = C^d \]

\[ H_2(2) = H_1 \otimes H_2, M_d^2(\mathbb{C}) \cong M_d(\mathbb{C}) \otimes M_d(\mathbb{C}) \text{ (isomorphic)} \]

Separable density matrices:

\[ \rho = \sum_{i,j} \lambda_{ij} P_{\psi_i} \otimes P_{\psi_j}, \sum_{i,j=1} \lambda_{ij} = 1 \text{ and } 0 \leq \lambda_{i,j} \leq 1 \]

(linear convex combination of the \( P_{\psi_i} \otimes P_{\psi_j} \), where \( P_{\psi_i} \) belongs to the first system and \( P_{\psi_j} \) belongs to the second system).

Example:

\[ |00><00| = |0><0| \otimes |0><0| = P_0^1 \otimes P_0^2 \]
\[ |11><11| = |1><1| \otimes |1><1| = P_1^1 \otimes P_1^2 \]

Non separable density matrices, those which cannot be written \( \sum_{i,j} \lambda_{ij} P_{\psi_i} \otimes P_{\psi_j} \).  

\[ |00><11| = |0><1| \otimes |0><1| \text{ cannot be written like } P_{\psi_i} \otimes P_{\psi_j}. \]

Transposition on \( M_2(\mathbb{C}) \) (2 \( \times \) 2 matrices)

\( T : M_2(\mathbb{C}) \rightarrow M_2(\mathbb{C}) \) it is a linear map which gives the transpose of a matrix belongs to \( M_2(\mathbb{C}) \)

\[ T \left[\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right] = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \]

Definition (positive map): A linear map \( \Lambda : M_d(\mathbb{C}) \rightarrow M_d(\mathbb{C}) \) is called positive iff any positive \( A \) is mapped in to positive \( \Lambda(A) \).

The matrix is positive iff all eigenvalues are positive.

Transposition is positive map since it doesn’t change the eigenvalues.

- \( T \) is positive map
- \( T^1 = T \otimes \text{id} \) : transposing the first factor and do nothing on the second factor.
\[ T^1 = T \otimes id: M_d(C) \otimes M_d(C) \to M_d(C) \otimes M_d(C) \] : partial transposition on the first factor.

\[
T^1[|0><0| \otimes |0><0|] = T^1 \left[ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\]

\[
T^1[|1><1| \otimes |1><1|] = T^1 \left[ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

\[
T^1[|0><1| \otimes |0><1|] = T^1 \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

\[
T^1[|1><0| \otimes |1><0|] = T^1 \left[ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

The partial transposition \((T^1)\) maps separable \(\rho\)'s into separable \(\rho\)'s.

\[
T^1[\rho] = T^1 \left[ \sum_{i,j} \lambda_{ij} P_{\psi^i_j} \otimes P_{\psi^j_i} \right]
\]

since transposition is linear

\[
T^1[\rho] = \sum_{i,j} \lambda_{ij} T[P_{\psi^i_j}] \otimes P_{\psi^j_i}, T[P_{\psi^i_j}]
\]

is also projector like \(P_{\psi^i_j}\).

Partial transposition behave gently to separable \(\rho\)'s but behave badly \(\rho\)'s which are non-separable. (gently means, don’t change the positiveness of \(\rho\)).

\[
T^1[|\beta_00><\beta_00|] = \frac{1}{2} \left[ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right] = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = X
\]
This is the partial transposition of $|\beta_{00}><\beta_{00}|$ and it is not density matrix.

$$X \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$X$ has negative eigenvalue.

Partial transposition isn’t a physical process. Because it only preserve the positivity in separable $\rho$’s.

The state $\rho$ on $C^2 \otimes C^2$ is entangled iff, after partial transposition $\rho$ become non-positive (has negative eigenvalue).

For entangled two system, unitary operation on one of them don’t produce the negative eigenvalue on to the other one, but non-unitary operation on the one of them can produce the negative eigenvalue on to the other one.

$$X = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

eigenvalues and their corresponding eigenvectors are (apart from the normalization),

$$\frac{1}{2} \rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \frac{1}{2} \rightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{2} \rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, -\frac{1}{2} \rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$|0> = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |1> = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ standard basis or computational basis.

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_x|0> = |1>, \sigma_x|1> = -|0>$$
$$\sigma_y|0> = i|0>, \sigma_y|1> = -i|1>$$
$$\sigma_z|0> = -|1>, \sigma_z|1> = |0>$$

where $u = 0, 1$

$\sigma_x$ makes the flip operation on the computational basis vectors.
Hadamard Transformation

\[
H |0> = \frac{1}{\sqrt{2}} (|0> + |1>) = |+>
\]
\[
H |1> = \frac{1}{\sqrt{2}} (|0> - |1>) = |->
\]
\[
H |1> = |0> + (-1)^x |1> (x = 0, 1)
\]

\[
H_{ij} = <i|H|j>
\]
\[
H = \begin{pmatrix}
<0|H|0> & <0|H|1>
\end{pmatrix}
\begin{pmatrix}
<1|H|0> & <1|H|1>
\end{pmatrix}
= \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 1
1 & -1
\end{pmatrix}
\]
in standard basis.

\[
H = H^\dagger
\]
\[
H^2 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}
= \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

\[
Sp(H) = \{1, -1\} \quad (Sp(H) \text{ denotes the spectrum of } H, \text{ i.e. the set of eigenvalues})
\]

\[
(\sigma_x - 1)(\sigma_x + 1) = 0. \quad \text{(from the quantum mechanics textbooks)} \quad \text{roots of the polynomial equation } (a - 1)(a + 1) = 0 \text{ gives the eigenvalues of the operator } \sigma_x.
\]

If the matrix is positive then it is self-adjoint.

\[
A = \begin{pmatrix}
1 \\ \alpha \\ 2
\end{pmatrix}
\]
we cant say the matrix A is positive or not.

For positiveness:

- \(Tr(A) \geq 0\) (sum of the eigenvalues must be greater than zero)
- \(Det(A) \geq 0\) (product of the eigenvalues must be greater than zero)

\[
Tr(A) = 3 \geq 0 \quad \text{and} \quad Det(A) = 2 - |\alpha|^2 \geq 0 \Rightarrow |\alpha|^2 \leq 2
\]

**CNOT Gate (Controlled-NOT)**

\[
\text{RESIMMM}
\]

Input is : \(|x> \otimes |y>\) (first bit \(|x>\) called control bit and the second one \(|y>\) is called target bit.)

Output is : \(|x> \otimes |x \oplus y>\) (\(\oplus\) denotes the summation in mod 2)
The gate implements:

- id operation if $|x > = |0 >$
- flip operation if $|x > = |1 >$

on the target bit.

*Machine that produces Bell States:*

**RESİM**

Incoming state: $|x > \otimes |y >$

$$H|x > = \frac{1}{\sqrt{2}} [|0 > + (-1)^x |1 >]$$

$H|x > \otimes |y >$, $H$ acts on $|x >$ only. It’s matrix representation must be $4 \times 4$. $H \otimes I$, then $H$ acts on $|x >$ and $I$ acts on $|y >$.

$$(H \otimes I) \left( |x > \otimes |y > \right) = \frac{1}{\sqrt{2}} [|0 > + (-1)^x |1 >] \otimes |y > = \frac{1}{\sqrt{2}} [|0 > \otimes |y > + (-1)^x |1 > \otimes |y > ]$$

**CNOT**: $\frac{1}{\sqrt{2}} [|0 > \otimes |y > + (-1)^x |1 > \otimes |1 + y > ] = |\beta_{xy} >$

$H$ has eigenvalues $\pm 1$ but this doesn’t mean that $H$ is not physical. It is an operation and preserves the positiveness of the matrix.

$$|\beta_{xy} > = \frac{1}{\sqrt{2}} [|0 > \otimes |y > + (-1)^x |1 > \otimes |1 + y > ]$$

$$|\beta_{00} > = \frac{1}{\sqrt{2}} [|0 > \otimes |0 > + |1 > \otimes |1 > ] = \frac{1}{\sqrt{2}} [|00 > + |11 > ]$$

$$|\beta_{01} > = \frac{1}{\sqrt{2}} [|0 > \otimes |1 > + |1 > \otimes |0 > ] = \frac{1}{\sqrt{2}} [|01 > + |10 > ]$$

$$|\beta_{10} > = \frac{1}{\sqrt{2}} [|0 > \otimes |0 > - |1 > \otimes |1 > ] = \frac{1}{\sqrt{2}} [|00 > - |11 > ]$$

$$|\beta_{11} > = \frac{1}{\sqrt{2}} [|0 > \otimes |1 > - |1 > \otimes |0 > ] = \frac{1}{\sqrt{2}} [|01 > - |10 > ]$$

These are the Bell States. ($|\beta_{11} >$ is the singlet state in quantum mechanics)

(Prove that Bell States are bases, i.e. they are orthonormal and span the 4-dimensional Hilbert space)

**Summary of the previous lecture:**

$|\beta_{00} >$ : entangled to qubit state.

$|\beta_{00} > = \frac{1}{\sqrt{2}} (|00 > + |11 >)$

Projection on to $|\beta_{00} >$ : $|\beta_{00} >< \beta_{00} |$

$$(T \otimes id)(|\beta_{00} >< \beta_{00} |) = X (X$ is not positive while $|\beta_{00} >$ is positive)$

Then $|\beta_{00} >$ is an entangled state.
\[ |\beta_{00} \rangle \rangle \beta_{00} | = \frac{1}{2} [P_0 \otimes P_0 + P_0 \otimes P_0 + |0 \rangle \rangle 1 | \otimes |0 \rangle \rangle 1 + |1 \rangle \rangle 0 | \otimes |1 \rangle \rangle 0 | \]

(First two terms are projections and last two are the interference terms)

One qubit state: \( |\psi > = \alpha |0 > + \beta |1 > \)

Observable \( \sigma_z \) : \[
\begin{cases} 
\text{eigenvalue } +1, & |0 > \text{ with probability } |\alpha|^2 \\
\text{eigenvalue } -1, & |1 > \text{ with probability } |\beta|^2 
\end{cases}
\]

interpretation: with probability \( |\alpha|^2 \) the system is in the state \( |0 > \) and with probability \( |\beta|^2 \) in the state \( |1 > \)

\[ \rho = |\alpha|^2 |0 \rangle \rangle 1 | + |\beta|^2 |1 \rangle \rangle 1 | 

but,

\[ |\psi \rangle \rangle |\psi \rangle = |\alpha|^2 |0 \rangle \rangle 1 | + |\beta|^2 |1 \rangle \rangle 1 | + \alpha \beta |0 \rangle \rangle 1 | + \bar{\alpha \beta} |1 \rangle \rangle 0 | 

last two terms are the interference terms and the interference terms prevents the statistical interpretation.

RESİMM

When only one of the two slit is open:

\[ |\alpha|^2 |0 \rangle \rangle 1 | + |\beta|^2 |1 \rangle \rangle 1 | 

But when two slits are open:

\[ |\alpha|^2 |0 \rangle \rangle 1 | + |\beta|^2 |1 \rangle \rangle 1 | + \alpha \beta |0 \rangle \rangle 1 | + \bar{\alpha \beta} |1 \rangle \rangle 0 | 

QUANTUM TELEPORTATION

It has been done for one qubit state (one photon).

Entanglement is obviously necessary.

Alice has one qubit \( |\psi > = \alpha |0 > + \beta |1 > \) and she shared an entangled state with Bob \( |\beta_{00} > = \frac{1}{\sqrt{2}} (|00 > + |11 >) \). (Two qubit system’s one qubit is on the Alice’s side (earth) and the other is on the Bob’s side (moon)) Alice wants to send her qubit to Bob.

Two qubits are on the Alice’s side.

One qubits are on the Bob’s side.

Alice acts locally (means that Alice make some operations only on her qubit), the operations are CNOT and measurement (her qubit) in the standard basis.
After getting outcome of the measurement she send the outcome to Bob with some classical communication channel. Then Bob will make the state $|\psi>$. 

RESİMМ

LOCC : Local operation and classical communication channel. (The distance in the experiment is one or two meter (distance between Alice and Bob))

Initial state:

$|\psi> \otimes |\beta_{00}> = \frac{1}{\sqrt{2}} [\alpha|0> + \beta|1>][|00> + |11>]$

CNOT : $|\psi'> = \frac{1}{\sqrt{2}} [\alpha(|000> + |011>) + \beta(|100> + |111>)]$

operation is local, there is nothing to do with Bob’s qubit.

HADAMARD : $|\psi'>\rightarrow |\psi''>$

$|\psi'> = \frac{1}{\sqrt{2}} [\alpha|0> \otimes (|000> + |011>) + \beta|1> \otimes (|100> + |011>)]$

$|\psi''> = \frac{1}{\sqrt{2}} [\alpha(|0> + |1>) \otimes (|000> + |011>) + \beta(|0> - |1>) \otimes (|100> + |011>)]$

Now Alice’s qubit state $|\psi>$ is on the Bob’s side (the first term of the linear superposition).

$|\psi''> = \frac{1}{\sqrt{2}} [|00> \otimes |\psi> + |01> \otimes \sigma_x |\psi> + |10> \otimes \sigma_z |\psi> + |11> \otimes (-i\sigma_y) |\psi> ]$

Now, measurement in the standard basis. After the Alice’s measurement on the final state $|\psi''>$,

<table>
<thead>
<tr>
<th>outcome</th>
<th>state after the measurement</th>
<th>what will Bob do on his qubit</th>
</tr>
</thead>
<tbody>
<tr>
<td>00</td>
<td>$</td>
<td>11&gt; \otimes</td>
</tr>
<tr>
<td>01</td>
<td>$</td>
<td>11&gt; \otimes \sigma_x</td>
</tr>
<tr>
<td>10</td>
<td>$</td>
<td>11&gt; \otimes \sigma_z</td>
</tr>
<tr>
<td>11</td>
<td>$</td>
<td>11&gt; \otimes (-i\sigma_y)</td>
</tr>
</tbody>
</table>

with the phone call, Bob knows the outcome and make some appropriate operation on his qubit for reaching the desired state $|\psi>$. 

Quantum mechanics is non-local but this does’nt violate the Relativity, because we cannot see this non-locality for superluminal signals.
Quantum Teleportation : 1 qubit with 2 cbits by LOCC

Dense Coding : 2 cbits with 1 qubit.

The necessity of the phone call in quantum teleportation is explained by partial trace.

\[ A = A^\dagger, \ E_\psi (A) = <\psi|A|\psi> = Tr (|\psi><\psi|A) \]

\[ \rho = \sum_{j=1}^{m} \lambda_j |\psi_j><\psi_j| \]

\[ E_\rho (A) = \sum_{j=1}^{m} \lambda_j E_\psi (A) = \sum_{j=1}^{m} \lambda_j Tr (|\psi_j><\psi_j|A) \]

since \( Tr \) operation is linear then,

\[ E_\rho (A) = Tr \left( \sum_{j=1}^{m} \lambda_j |\psi_j><\psi_j|A \right) \]

\[ \Rightarrow E_\rho (A) = Tr (\rho A) \]

\( \rho \)'s are the most general states. When \( \rho = |\psi_j><\psi_j| \) (i.e. pure state), then \( Tr (|\psi_j><\psi_j|A) \) gives the mean value of \( A \) on the pure state.

Final state of the quantum teleportation : \( |\psi''> \) then \( \rho'' = |\psi''><\psi''| \)

\( H_A \otimes H_B \) : Hilbert space for the whole system, where \( H_A \) is the Hilbert space of the Alice’s qubits, \( H_B \) is the Hilbert space of the Bob’s qubit.

\( A \otimes B \) : Observable on the whole system, where \( A \) is the observable for the Alice’s qubits, \( B \) is the observable for the Bob’s qubit.

Can Bob perform measurement on the moon which can learn something what’s going on on the earth?

On the moon : \( 1 \otimes B \), \( E_{\rho''}(1 \otimes B) = Tr (|\psi''><\psi''|1 \otimes B) \)

Alice’s Hilbert space : \( H_A, \{|\psi_A^i>\} \) ONB

Bob’s Hilbert space : \( H_B, \{|\psi_B^i>\} \) ONB
Then for $H_A \otimes H_B \{ |\psi^i_A > \otimes |\psi^j_B > \}$ is ONB.

$$Tr (|\psi'' > < \psi''| 1 \otimes B) = \sum_{i,j} (\langle \psi^i_A | \otimes < \psi^j_B | \psi'' > < \psi''| 1 \otimes B |\psi^j_B >)$$

$$= \sum_{i,j} < \psi^i_A | \otimes < \psi^j_B | \psi'' > < \psi''| 1 \otimes B |\psi^j_B >)$$

$$= \sum_{i,j} < \psi^i_B | \rho''_2 | \psi^j_B >$$

where

$$\rho''_2 = \sum_{i} < \psi^i_A | \psi'' > < \psi''| \psi^i_A >$$

(Partial trace operation, $Tr_1 [\rho''_2]$)

$$E_{\rho''} (1 \otimes B) = E_{\rho''_2} (B)$$

($\rho''_2$ is the system on the Bob’s side and the $\rho''$ is the global system)

$$\rho''_2 = <00|\rho''|00> + <01|\rho''|01> + <10|\rho''|10> + <11|\rho''|11>$$

This eliminates the degrees of freedom of Alice.

If $\rho''_2$ hasn’t sign on what’s going on earth then there is no sperluminal communication.

After calculations,

$$\rho''_2 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{(tracial state)}$$

$$S (\rho''_2) = \ln 2$$

Then, Bob has to wait to phone call, because $\rho''_2$ doesn’t contain information on the what’s going on on the earth (Alice’s side).

This was for the time after the Hadamard transformation and before the measurement.

If calculation were made for the time after the measurement and before the phone call, $(M_{xy} [\psi'' > < \psi''])$

$$Tr_1 [M_{xy} (|\psi'' > < \psi'')] \text{ then } \rho''_2 \text{ will be again tracial state. (prove?)}$$

Then after or before the measurement Bob doesn’t know anything about the what’s going on on the earth. Thus Alice have to send the outcomes with some classical communication channel.

**DENSE CODING**

Alice wants to send Bob 2 classical bits. Before starting they agree the following protocol:
- If Bob gets $|\beta_{00} > = \frac{1}{\sqrt{2}} (|00 > + |11 >)$ then he writes 00
- If Bob gets $|\beta_{01} > = \frac{1}{\sqrt{2}} (|01 > + |10 >)$ then he writes 01
- If Bob gets $|\beta_{10} > = \frac{1}{\sqrt{2}} (|00 > - |11 >)$ then he writes 10
- If Bob gets $|\beta_{11} > = \frac{1}{\sqrt{2}} (|01 > - |10 >)$ then he writes 11

Alice and Bob sharing $|\beta_{00} >$ means Alice has one qubit, Bob has one qubit and the whole system (two qubit) is in the state $|\beta_{00} >$. (Third person -Charlie- using the machine which produces these Bell States and send them to Alice and Bob)

In the sharing state $|\beta_{00} >$, Alice could send her qubit to Bob and then the state is only on the Bob’s side (not shared anymore).

For sending 01 cbit she has to convert $|\beta_{00} >$ to $|\beta_{01} >$ local means.

- For cbit 00, $1 \otimes 1|\beta_{00} > \rightarrow |\beta_{00} >$
- For cbit 01, $\sigma_x \otimes 1|\beta_{00} > \rightarrow |\beta_{01} >$
- For cbit 10, $\sigma_z \otimes 1|\beta_{00} > \rightarrow |\beta_{10} >$
- For cbit 11, $-i\sigma_y \otimes 1|\beta_{00} > \rightarrow |\beta_{11} >$

and then she send her qubit to Bob.

<table>
<thead>
<tr>
<th>state to be wanted to send</th>
<th>Alice’s qubit</th>
<th>Bob’s qubit</th>
</tr>
</thead>
<tbody>
<tr>
<td>00</td>
<td>$</td>
<td>\beta_{00} &gt;$</td>
</tr>
</tbody>
</table>
| 01                        | $\sigma_x|\beta_{00} >$ | $|\beta_{01} >$ | Bob, when take
| 10                        | $\sigma_z|\beta_{00} >$ | $|\beta_{10} >$ |
| 11                        | $-i\sigma_y|\beta_{00} >$ | $|\beta_{11} >$ |

Alice’s qubit, has to measure in Bell state.

Quantum mechanics can do a lot of things which classical mechanics cannot do. But something exist which are classical mechanics can do but quantum mechanics cannot (e.g. copy).

The No-Cloning Theorem

To reproduce the $|\psi >$ state

$$|\psi > \otimes |0 > \rightarrow |\psi > \otimes |\psi > = U (|\psi > \otimes |0 >)$$ (1)

$$|\phi > \otimes |0 > \rightarrow |\phi > \otimes |\phi > = U (|\phi > \otimes |0 >)$$ (2)
$U$ is copier. By taking the bra of the last expression in (1) and the ket of the last expression in (2)

\[
(<\psi| \otimes <0|U^\dagger) (U|\phi > \otimes |0>) = <\psi| \otimes <0|U^\dagger U|\phi > \otimes |0>
\]

\[
= (<\psi| \otimes <0|) (|\phi > \otimes |0>)
\]

\[
= <\psi|\phi > <0|0> = <\psi|\phi >
\]

and by making the same product with, middle expression of (1) and (2)

\[
(<\psi| \otimes <\psi|) (|\phi > \otimes |\phi>) = <\psi|\phi > <\psi|\phi >
\]

\[
= (<\psi|\phi >)^2
\]

But it has to equal to $<\psi|\phi >$ i.e.

\[
<\psi|\phi > = (<\psi|\phi >)^2
\]

This is possible only for $<\psi|\phi > = 0$ or $<\psi|\phi > = 1$.

This means that quantum copier can copy only the orthogonal states.

For the basis $|\psi_1 >$ and $|\psi_2 >$, quantum copier can copy these two but not $|\psi > = \alpha |\psi_1 > + \beta |\psi_2 >$

summary of the last lecture

No-Cloning Theorem

\[
|\psi > \otimes |0> \rightarrow |\psi > \otimes |\psi >
\]

\[
|\phi > \otimes |0> \rightarrow |\phi > \otimes |\phi >
\]

iff $<\psi|\phi > = 0$ or $<\psi|\phi > = 1$.

We cannot do the cloning (for non-orthogonal states) with the same machine $(U)$.

\[
<\tilde{\psi}|\psi > \neq 0 \Rightarrow U|\tilde{\psi} > \otimes |0> = |\tilde{\psi} > \otimes |\tilde{\psi} >
\]

is impossible.

There is no quantum copy machine for all possible states.

Teleportation:

\[
\begin{align*}
\text{Alice} & \rightarrow \text{Bob} \\
|\psi > & \rightarrow |\psi >
\end{align*}
\]

There is no copy in teleportation because the state is on the Alice’s side is disappear and appear on the Bob’s side.

Comment on possibility of the cloning the states:
\[ < \phi | \psi > = 1, \text{ Cauchy-Schwartz } | < \phi | \psi > | \leq \| \phi \| \| \psi \|. \]
Both side of the inequality is 1 due to \( < \phi | \psi > = 1 \) and normed states (\( | \phi >, | \psi > \)). This means that \( | \phi > \) and \( | \psi > \) is the same state.

\[
U | \psi > \otimes | 0 > \quad \rightarrow \quad < \psi | \otimes < 0 | U^\dagger
\]
\[
= | \psi > \otimes | \psi > \quad \rightarrow \quad < \psi | \otimes < \psi |
\]
ket \quad \rightarrow \quad bra

Since \( U \) is unitary (\( U = U^\dagger \)) then,

\[
U^\dagger | \psi > \otimes | \psi > = | \psi > \otimes | 0 >
\]
this is the transition from the cloning to erasing.

\[
| \psi > \in C^d , \quad U : C^d \otimes C^d \rightarrow C^d \otimes C^d
\]
\[
| 0 > \in C^d , \quad U (| \psi > \otimes | 0 >) : \text{ } U \text{ acts not only on the second qubit. It acts on the whole system, it sees the first qubit (|} \psi > \text{) and copying it into second one (|} 0 > \rightarrow |} \psi > \text{).}
\]

**TACKLING ENTANGLEMENT**

There is no settle theory of entanglement.

Review : quant-ph/0109124

Partial Transposition : \( T : \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \)

transposition is positive map : \( T \) preserves the positivity, and it is linear.

(if \( X \in M_d (C) \) and \( X \geq 0 \Rightarrow T [X] \geq 0 : \text{ positive map} \))

\( \Lambda : M_d (C) \rightarrow M_d (C) \) is positive iff \( X \geq 0 \Rightarrow \Lambda [X] \geq 0 \)

map on the matrices (observables) not on vectors.

\( M_d (C) : d \times d \text{ matrices.} \)

\( M_2 (C) \otimes M_2 (C) \rightarrow M_2 (C) \otimes M_2 (C) : T_1 = T \otimes id \) (partial transposition)

\( T \) is positive and \( id \) is positive \( \Rightarrow T \otimes id \) is positive ? NO!

\( X \in M_2 (C) \otimes M_2 (C) , X = | \beta_{00} > < \beta_{00} | \)

\( X \geq 0 \text{ iff } < \psi | X | \psi > \geq 0 \text{ for all } | \psi > \)

\[
< \psi | \beta_{00} > < \beta_{00} | \psi > = | < \beta_{00} | \psi > |^2 \geq 0
\]
\[ X = |\beta_{00} \rangle \langle \beta_{00}| : \text{projector on the state } |\beta_{00} \rangle . \text{ It’s eigenvalues are 0 and 1.} \]

\[ T_1 [|\beta_{00} \rangle \langle \beta_{00}|] \neq 0 \]

**COMPOUND SYSTEM (BIPARTITE SYSTEM)**

\[ C^d \otimes C^d = C^{d^2} \text{ (Alice and Bob)} \]

\[ M_d(C) \otimes M_d(C) \cong M_{d^2}(C) \text{ (isomorphic, same type algebra)} \]

Separable states : \( \rho = \sum_{i,j} \lambda_{ij} P^1_i \otimes P^2_j \) where \( P^1_i \) is Alice’s projection and \( P^2_j \) Bob’s projection, \( 0 \leq \lambda_{ij} \leq 1 \)

Inseparable states (entangled states) are which cannot be written like above.

\[ \tau = \frac{1}{4} \left( \begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{array} \right) \]

state is separable.

\[ \tau = \frac{1}{4} \mathbf{1}, \{ |j \rangle \}_{1}^{4} \text{ ONB in } C^4, P_j = |j \rangle \langle j| \text{ and } \sum_j P_j = \mathbf{1} \]

\[ \Rightarrow \tau = \frac{1}{4} \sum_{j=1}^{4} |j \rangle \langle j| \]

\[ |1 \rangle = |0 \rangle \otimes |0 \rangle, |2 \rangle = |0 \rangle \otimes |1 \rangle, |3 \rangle = |1 \rangle \otimes |0 \rangle, \]

\[ |4 \rangle = |1 \rangle \otimes |1 \rangle \]

\[ \Rightarrow \tau = \frac{1}{4} \sum_{j=1}^{4} |xy \rangle \langle xy| = \frac{1}{4} \sum_{j=1}^{4} |x \rangle \langle x| \otimes |y \rangle \langle y| \]

Another ONB in \( C^4 \) is Bell basis.

\[ \tau = \frac{1}{4} \sum_{j=1}^{4} |\beta_{xy} \rangle \langle \beta_{xy}| \]

this seems entangled (inseparable) because it is linear superposition of entangled basis vectors. But this is not true. \( \tau \) is separable as shown above.

*Bipartite 2-qubit system*
Alice and Bob has one qubit. Then 2-qubit system is in $C^2 \otimes C^2$ and observables are in $M_2(C) \otimes M_2(C)$. The question is: if $\rho$ is a state of the $A + B$ when is $\rho$ entangled (inseparable)?

This question is hard to answer.

(Ashar Peres) $\rho_{AB}$ is separable iff $T_1[\rho_{AB}] \geq 0$ or $T_A \otimes id$ preserves of the positivity of $\rho$. (For only two qubit system)

In general: (in dimension $d$) suppose $\rho$ is separable i.e. $\rho = \sum_{i,j} \lambda_{ij} P^1_i \otimes P^2_j \in M_d(C) \otimes M_d(C)$

$$T_1[\rho] = T \otimes id[\rho] = [T \otimes id]\left[\sum_{i,j} \lambda_{ij} P^1_i \otimes P^2_j \right]$$

since $T$ is linear then

$$T_1[\rho] = \sum_{i,j} \lambda_{ij} [T \otimes id]\left[P^1_i \otimes P^2_j \right] = \sum_{i,j} \lambda_{ij} T[P^1_i] \otimes P^2_j$$

the two factor in the tensor product are positive. (Because transposition and identity don’t change the eigenvalues)

If we have $T[P]$ this also projection.

$$P = |\psi \rangle \langle \psi|,$$ in some basis

$$P = \begin{pmatrix} |\psi_1|^2 & \psi_1 \bar{\psi}_2 & \ldots & \psi_1 \bar{\psi}_d \\ \psi_2 \bar{\psi}_1 & |\psi_2|^2 & \ldots & \psi_2 \bar{\psi}_d \\ \vdots & \vdots & \ddots & \vdots \\ \psi_d \bar{\psi}_1 & \psi_d \bar{\psi}_2 & \ldots & |\psi_d|^2 \end{pmatrix}$$

$$\Rightarrow P^T = \begin{pmatrix} |\bar{\psi}_1|^2 & \bar{\psi}_1 \psi_2 & \ldots & \bar{\psi}_1 \psi_d \\ \bar{\psi}_2 \psi_1 & |\bar{\psi}_2|^2 & \ldots & \bar{\psi}_2 \psi_d \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\psi}_d \psi_1 & \bar{\psi}_d \psi_2 & \ldots & |\psi_d|^2 \end{pmatrix}$$

with the basis

$$|\psi^*\rangle = \begin{pmatrix} \bar{\psi}_1 \\ \bar{\psi}_2 \\ \vdots \\ \bar{\psi}_d \end{pmatrix}$$

$$\Rightarrow <\psi^*| = \begin{pmatrix} \psi_1 & \psi_2 & \ldots & \psi_d \end{pmatrix}$$

$P^T$ becomes projection again $P^T = |\psi^* \rangle \langle \psi^*|$
While $P$ projects the states on $|\psi>\rangle$, $T[P]$ projects the states on $|\psi^*>\rangle$.

$T \geq 0$, $T \otimes id$ preserves positivity of separable states.

$\Lambda \geq 0$, $\Lambda \otimes id$ preserves positivity of separable states. (where $\Lambda$ is general operator)

But although $T[P]$ is also projector, we cannot say whether $\Lambda[P]$ is again projector.

But suppose that $T_1[\rho] \neq 0$ then is $\rho$ inseparable? (i.e. the reverse of the statement which is given above is true?)

The answer is :

- YES in $M_2(C) \otimes M_2(C)$.
- NO in $M_d(C) \otimes M_d(C)$ where $d \geq 3$

The signs of the entanglement in $d \geq 3$ is still an open question.

A state $\rho$ is called PPT (positive partial transposed state) if $T_1[\rho] > 0$. (i.e. $T_1$ is positive operator)

- if $d = 2$ then PPT states are separable.
- if $d > 2 \exists$ PPT states which are entangled (inseparable) (This is called bound entanglement)

$\rho$ separable $\Rightarrow T_1[\rho] \geq 0$

But $T_1[\rho] \geq 0 \Rightarrow \rho$ separable

\[
\begin{cases} 
\text{YES} & \text{in } M_2(C) \otimes M_2(C) \\
\text{NO} & \text{in } M_d(C) \otimes M_d(C) \text{ where } d \geq 3 
\end{cases}
\]

**Distillation of Entanglement**

$|\beta_{00}> = \frac{1}{\sqrt{2}} (|00> + |11>)$: maximally entangled state

Environment, heat bath, reservoir$\leftrightarrow$ quantum/classical noise (DECOHERENCE)

The machine that produce Bell states is in environment of course then the noise.
Due to decoherence,

\[ |\psi| = \cos \theta |00| + \sin \theta |11| \]

where \( \cos \theta \neq \sin \theta \). \( |\psi| \) represents imperfections of the preparation of \( |\beta_{00}| \).

Maximally Entangled States are of the form \( U_1 \otimes U_2 |\beta_{00}| \) where \( U_{1,2} \) are unitaries. \( |\beta_{00}| \) is maximally entangled state and the other maximally entangled states can be obtained by performing unitary operations on \( |\beta_{00}| \).

### Distillation Of Entanglement:

\[ |\psi| \otimes m = |\psi| \otimes |\psi| \otimes ... \otimes |\psi| \quad \text{ (number of } m \text{ } |\psi| \text{)} \]

\[
\begin{array}{c|c}
\text{number of } m \text{ } & \text{pure state (not maximum entangled)} \rightarrow \text{pure state (maximum entangled)} \\
\hline
|\psi| \rightarrow |\beta_{00}| & |\beta_{00}| \\
|\psi| \rightarrow |\beta_{00}| & |\beta_{00}| \\
\vdots & \vdots \\
|\psi| \rightarrow |\beta_{00}| & |\beta_{00}| \\
\end{array}
\]

\( n(m) \)

\[ \varphi_A : \rho \rightarrow \varphi_A[\rho] \text{ Alice’s local operation} \]

\[ \varphi_B : \rho \rightarrow \varphi_B[\rho] \text{ Bob’s local operation} \]

A, B bipartite LOCC. Local operation : \( \varphi_A \otimes \varphi_B \)

Take \( m \) copy of \( |\psi| \) (not maximally entangled) and with local operation convert them \( |\beta_{00}| \). \( |\beta_{00}| \) are known singlets

\[ \frac{n(m)}{m} \equiv \frac{\text{number of maximum entangled}}{\text{number of not maximum entangled}} \]

\[ \lim_{m \to \infty} \frac{n(m)}{m} = E_D \text{ (distillation of entanglement)} \]

This procedure is called distillation of entanglement. (Distillation of \( |\beta_{00}| \) from \( |\psi| \))

\[ |\psi| = \cos \theta |00| + \sin \theta |11| \quad (\cos \theta \neq \sin \theta) \]

\[ |\beta_{00}| = \frac{1}{\sqrt{2}} (|00| + |11|) \]

By noise the pure state can becomes mixed state.

\text{NOISE : } |\psi| <\psi| \rightarrow N[|\psi| <\psi|] \quad (N \text{ stands for the noise and resulting state is mixture})

\[ N[|\psi| <\psi|] = \sum_1^i \lambda_i P_i \]
This is also known \textit{DECOHERENCE}.

Due to the noise preparation of state may not give the state $|\psi> = \cos \theta |00> + \sin \theta |11>$, instead of this preparation gives the mixtures.

\[
\rho \otimes \rho \otimes ... \otimes \rho \xrightarrow{\text{LOCC}} \text{number of } m \left\{ \begin{array}{c} |\beta_{00}> \\ |\beta_{00}> \\ ... \\ |\beta_{00}> \end{array} \right\}
\]

Can we distillate from mixtures to maximum entangled states?

Answer is: NO!, if $\rho$ is entangled and PPT.

We cannot distill $|\beta_{00}>$ from bounded entangled state. We cannot create by LOCC from separable states to entangled states for bipartite systems.

If we have tiny entanglement and PPT we cannot amplify it to maximum entangled state for $d \geq 3$.

**SUMMARY**

- PPT
- $M_2(C) \otimes M_2(C) : \text{PPT} \leftrightarrow \text{separability (no entanglement)}$
- $M_d(C) \otimes M_d(C) : d \geq 3 \exists \text{ PPT states which are entangled (bound entanglement)}$
- Bound entanglement cannot be distilled.

Bound entanglement cannot using for teleportation. Free entanglement can using for teleportation.

\textit{Thermodynamics of Entanglement}:

\[
E_{\text{total}} = E_{\text{free}} + E_{\text{bound}}
\]

Distillation: number of $m$ $|\psi> \rightarrow$ number of $n(m) |\beta_{00}>$

\[
\lim_{m \rightarrow \infty} \frac{n(m)}{m} = E_D
\]
Dilution (Formation) : number of $n (m) \ |\beta_{00} >$ number of $m \ |\psi >$

$$\lim_{m\to\infty} \frac{n(m)}{m} = E_F$$

Out of number of $m$ not maximally entangled state $\rho$ we have protocol to get
number of $n (m)$ maximally entangled state $|\beta_{00} >$.

The reverse of this procedure is also possible (namely dilution, formation).

$|\psi_{AB} >$ pure state

$$Tr_B (|\psi_{AB} > < \psi_{AB} |) = \rho_A$$
$$Tr_A (|\psi_{AB} > < \psi_{AB} |) = \rho_B$$

For pure states $S (\rho_A) = S (\rho_B)$

$$E_D = E_F = S (\rho_A) = S (\rho_B)$$

$$|\beta_{00} >^{\text{formation}} \rho \xrightarrow{\text{distillation}} |\beta_{00} >^{\text{final}}$$

When we have bound entanglement, the formation and the distillation not be reversible. (Like thermodynamics, some things waste somewhere due to the $E_{\text{bound}}$.

$E_{\text{bound}}$ : heat, $E_{\text{free}}$ : work) For the part $E_{\text{free}}$ these processes are reversible.

**COMPLETE POSITIVITY**

**Schrodinger Equation :** $i \partial_t |\psi_t > = H |\psi_t > \quad (\hbar, m, ... = 1)$

Solution is : $|\psi_t > = e^{-iHt} |\psi > = U_t |\psi >$ where $U_t^\dagger U_t = U_t U_t^\dagger = 1$

$$i \partial_t \left( \begin{array}{c} a_t \\ b_t \end{array} \right) = \omega \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \left( \begin{array}{c} a_t \\ b_t \end{array} \right)$$

: rotation about $z$ with frequency $\omega$.

How density matrices evolve in time?

$$|\psi >\rightarrow |\psi_t > = U_t |\psi >$$
$$< \psi |\rightarrow < \psi_t | = < \psi |U_t^\dagger$$

$$\Rightarrow |\psi > < \psi |\rightarrow U_t |\psi > < \psi |U_t^\dagger$$

Then

$$\rho = \sum_i \lambda_i |\psi_i > < \psi_i | \rightarrow \rho_t = \sum_i \lambda_i U_t |\psi_i > < \psi_i |U_t^\dagger$$

$$= U_t \rho U_t^\dagger$$

$$\partial_t \rho_t = \partial_t \left( U_t \rho U_t^\dagger \right)$$

$$= \partial_t U_t \rho U_t^\dagger + U_t \rho \left( \partial_t U_t^\dagger \right)$$

$$= -iHU_t \rho U_t^\dagger + iHU_t \rho U_t^\dagger$$
(since $H$ and $U$ are commute)

Unitary quantum time evolution: $\partial_t \rho_t = -i [H, \rho_t]$

$\rho \rightarrow \rho_t = U_t \rho U_t^\dagger$ where $U = e^{-iHt}$

is $U_t [\rho]$ positive? (evolves $\rho$ into $U_t [\rho]$)

$\rho$ is positive then if $U_t [\rho]$ is positive then operation is positive

$\langle \psi | U_t [\rho] | \psi \rangle = \langle \psi | U_t \rho U_t^\dagger | \psi \rangle = \langle \psi_t | \rho | \psi_t \rangle \geq 0$

Beside this, time evolution has to be positive, since it is occur in the universe.

$U_t$ transforms pure states into pure states.

Then the noise: $\partial_t \rho_t = -i [H, \rho_t] + D [\rho_t]$

(Last term is due to the noise) with noise pure states $\rightarrow$ mixed states.

$\rho \rightarrow D [\rho]$ operation (linear map)

$\langle \psi |, A = A^\dagger = \sum_i a_i |a_i \rangle \langle a_i |$

getting probability $a_i$ is, $|\langle \psi | a_i \rangle|^2$ afterwards the system is in the state $|a_i >$

$\begin{align*}
a_1 & p_\psi (a_1) |a_1 \rangle \langle a_1 |
\end{align*}$

$\begin{align*}
a_2 & p_\psi (a_2) |a_2 \rangle \langle a_2 |
\end{align*}$

$\begin{align*}
\vdots & \vdots \vdots \vdots \vdots \vdots
\end{align*}$

$\begin{align*}
a_d & p_\psi (a_d) |a_d \rangle \langle a_d |
\end{align*}$

$p_\psi (a_i)$: frequency of the $a_i$ when one makes a lot of measurement on the system.

$|\psi \rangle < \psi | \rightarrow \sum_i p_\psi (a_i) |a_i \rangle \langle a_i |$ (from pure state to mixed state)

$M_A : |\psi \rangle < \psi | \rightarrow M_A [|\psi \rangle < \psi |] = \sum_{i=1}^d p_\psi (a_i) |a_i \rangle \langle a_i |$

$= \sum_{i=1}^d < \psi |a_i \rangle \langle a_i | \psi |a_i \rangle \langle a_i |$

$= \sum_{i=1}^d P_{a_i} |\psi \rangle < \psi | P_{a_i}$

Projective Measurement

$A = \sum_{i=1}^d a_i P_{a_i} \quad \leftrightarrow \quad M_A [|\psi \rangle < \psi |] = \sum_{i=1}^d P_{a_i} |\psi \rangle < \psi | P_{a_i}$

$\rho = \sum_j \lambda_j |\psi_j \rangle < \psi_j | \rightarrow \sum_j \lambda_j M_A [|\psi_j \rangle < \psi_j |] = \sum_i P_{a_i} \rho P_{a_i}$
Doing this kind of transformation on the state’s is projective measurement.

POVM (Positive Operator Valued Measurement)

\[ E_j : C^d \to C^d \quad \text{where} \quad \sum_{j=1}^{m} E_j^\dagger E_j = \mathbf{1} \]

POVM is any set of operators \( \{E_j\}_{j=1}^{m} \) which satisfies \( \sum_{j=1}^{m} E_j^\dagger E_j = \mathbf{1} \).

\[ \rho \to \varepsilon [\rho] = \sum_{j=1}^{m} E_j \rho E_j^\dagger \]

In particular case \( E_j = P_j \ (j = 1, 2, ..., d) \)

\[ \sum_{j=1}^{m} E_j^\dagger E_j = \sum_{j=1}^{m} P_j^\dagger P_j = \mathbf{1} \]

\( (P_j P_k = \delta_{jk} P_k) \)

Is trace preserved?

\[
\begin{align*}
\text{Tr} (\varepsilon [\rho]) &= \text{Tr} \left( \sum_{j=1}^{m} E_j \rho E_j^\dagger \right) \\
&= \sum_{j=1}^{m} \text{Tr} \left( E_j \rho E_j^\dagger \right) \\
&= \sum_{j=1}^{m} \text{Tr} \left( E_j^\dagger E_j \rho \right) \\
&= \text{Tr} \left( \sum_{j=1}^{m} E_j^\dagger E_j \rho \right) \\
&= \text{Tr} (\rho) \\
&= 1
\end{align*}
\]

(Since trace operation is linear and cyclic)

Is \( \varepsilon \) positive?

Prove that \( \varepsilon [\rho] = \sum_{j=1}^{m} E_j \rho E_j^\dagger \geq 0 \) \( (\sum_{j=1}^{m} E_j^\dagger E_j = \mathbf{1}) \).

Positivity : \( \forall \psi >, < \psi | \varepsilon [\rho] | \psi > \geq 0. \)

\[ < \psi | \varepsilon [\rho] | \psi > = < \psi \sum_{i=1}^{m} E_i \rho E_i^\dagger | \psi > \]
since scalar product is linear

$$< \psi | [\rho] | \psi > = \sum_{i=1}^{m} < \psi | E_i [\rho] E_i^\dagger | \psi >$$

if all of them $$< \psi | E_j [\rho] E_j^\dagger | \psi >$$ is positive then the sum is positive.

$$E_j^\dagger | \psi > = | \phi_j > \Rightarrow < \psi | E_j = < \phi_j$$

$$\Rightarrow < \psi | E_j [\rho] E_j^\dagger | \psi > = < \phi_j | [\rho] | \phi_j > \geq 0$$

then all terms in the sum are positive. Thus $$[\rho]$$ is positive i.e. $$\varepsilon$$ is positive.

Prove that $$A = A^\dagger \geq 0 \iff A = \sum_{i=1}^{m} a_i | a_i > < a_i |$$ with $$a_i \geq 0$$.

$$< \psi | A | \psi > \geq 0 \Rightarrow < \psi | A | \psi > = < \psi | \left( \sum_{i=1}^{m} a_i | a_i > < a_i | \right) | \psi >$$

$$= \sum_{i=1}^{m} < \psi | a_i > < a_i | \psi > a_i$$

Since this must be true for all $$| \psi >$$ then it must be true also for $$| \psi > = | a_j >$$ where $$< a_j | a_i > \delta_{ji}$$, $$\{|a_j >\}_{j=1}^{d}$$ ONB

$$\sum_{i=1}^{m} < \psi | a_i > < a_i | \psi > a_i = \sum_{i=1}^{m} < a_j | a_i > < a_i | a_j > a_i$$

$$= \sum_{i=1}^{m} \delta_{ji} a_i = a_j \geq 0$$

Now the proof of the reverse (since $$\iff$$)

A has positive eigenvalues then $$< \psi | \left( \sum_{i=1}^{m} a_i | a_i > < a_i | \right) | \psi >$$ is positive because $$\sum_{i=1}^{m} a_i | a_i > < a_i |$$ is positive.

Example:

$$\rho \in M_2 (C), \partial_t \rho_t = -i [\omega \sigma_z, \rho_t]$$ with $$H = \omega \sigma_z$$
Since $\rho_t$ is state we can represent it with $\rho_t = \begin{pmatrix} x_t & y_t \\ \bar{y}_t & 1 - x_t \end{pmatrix}$

$$\Rightarrow \partial_t \rho_t = \begin{pmatrix} \dot{x}_t & \dot{y}_t \\ \dot{\bar{y}}_t & -\dot{x}_t \end{pmatrix} = -i\omega (\sigma_z \rho_t - \rho_t \sigma_z)$$

$$= -i\omega \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_t & y_t \\ \bar{y}_t & 1 - x_t \end{pmatrix} - \begin{pmatrix} x_t & y_t \\ \bar{y}_t & 1 - x_t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

$$= -i\omega \left\{ \begin{pmatrix} x_t & y_t \\ -\bar{y}_t & -1 + x_t \end{pmatrix} - \begin{pmatrix} x_t & -y_t \\ \bar{y}_t & -1 + x_t \end{pmatrix} \right\}$$

$$= -i\omega \begin{pmatrix} 0 & 2y_t \\ -2\bar{y}_t & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \dot{x}_t & \dot{y}_t \\ \dot{\bar{y}}_t & -\dot{x}_t \end{pmatrix} = \begin{pmatrix} 0 & -2i\omega y_t \\ 2i\omega \bar{y}_t & 0 \end{pmatrix}$$

$$\Rightarrow \dot{x}_t = 0 \text{ and } \dot{\bar{y}}_t = 2i\omega \bar{y}_t$$

Initial conditions

$$\rho = \begin{pmatrix} x & y \\ \bar{y} & 1 - x \end{pmatrix}$$

$$\Rightarrow x_t = x \text{ and } \bar{y}_t = ye^{-2i\omega t}$$

$$\Rightarrow \rho_t = \begin{pmatrix} x & ye^{-2i\omega t} \\ \bar{y}e^{2i\omega t} & 1 - x \end{pmatrix}$$

$\rho \in M_2(C), \rho = \frac{1}{2} [1 + \alpha^\sigma \sigma^\alpha], \alpha = (a_1, a_2, a_3) \in \mathbb{R}^3 \text{ and } \sigma^\alpha = (\sigma_1, \sigma_2, \sigma_3)$ where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Under which conditions $\rho$ is a state?

$$Tr(\rho) = 1 \text{ and } Det(\rho) > 0 \text{ (for positiveness)}$$

Since $\sigma$’s are traceless then $Tr(\rho) = 1$

$$Det(\rho) \geq 0 \Rightarrow |\alpha| \leq 1$$

If $a = 1 \Rightarrow Det(\rho) = 0$ and $Tr(\rho) = 1$. This means that eigenvalues are 0 and 1 then it is pure state.

Given any matrix $2 \times 2, X = \sum_{\mu=0}^{3} \alpha^\mu \sigma_\mu, X \in M_2(C)$ where $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\sigma_i \sigma_j = \delta_{ij} \sigma_0 + i\varepsilon_{ijk} \sigma_k$$

$$X^i = \frac{1}{2} (Tr(\sigma_i X))$$

$$X^0 = \frac{1}{2} Tr(X)$$

$$X^\mu = \frac{1}{2} Tr(X \sigma_\mu)$$
\[ \partial_t \rho_t = -i [H, \rho_t], \quad H = \omega \sigma_z \]

\[ \rho_t = \frac{1}{2} 1 + \frac{1}{2} \rho^i(t) \sigma_i \] (Einstein summation convention)

\[ \rho_{t=0} = \rho = \left( \begin{array}{cc} x & y \\ \frac{y}{x} & 1-x \end{array} \right) \]

\[ \partial_t \rho_t = \frac{1}{2} \rho^i(t) \sigma_i = -i \omega \left[ \sigma_z, \frac{1}{2} 1 + \frac{1}{2} \rho^i(t) \sigma_i \right] \]

commutator is linear

\[ \Rightarrow \partial_t \rho_t = -i \omega \left[ \sigma_3, \frac{1}{2} 1 \right] - i \omega \left[ \sigma_3, \frac{1}{2} \rho^i(t) \sigma_i \right] \]

\[ \Rightarrow \frac{1}{2} \dot{\rho}^i(t) \sigma_i = -i \omega \left[ \rho^i(t) \sigma_i \right] \]

\[ \Rightarrow \dot{\rho}^i(t) \sigma_i = -i \omega \rho^i(t) [\sigma_3, \sigma_i] \]

\[ \Rightarrow \dot{\rho}^i(t) \sigma_i = -i \omega \left[ \rho^1(t) [\sigma_3, \sigma_1] + \rho^2(t) [\sigma_3, \sigma_2] \right] \]

\[ \dot{\rho}^i(t) \sigma_i = 2 \omega \rho^1(t) \sigma_2 - 2 \omega \rho^2(t) \sigma_1 \]

\[ \dot{\rho}^1(t) \sigma_1 + \dot{\rho}^2(t) \sigma_2 + \dot{\rho}^3(t) \sigma_3 = 2 \omega \rho^1(t) \sigma_2 - 2 \omega \rho^2(t) \sigma_1 \]

with equating the coefficients of the \( \sigma_i \)'s (if \( \sum_{\mu=0}^3 X^\mu \sigma_\mu = \sum_{\mu=0}^3 Y^\mu \sigma_\mu \) then all the coefficients must be equal \( X^\mu = Y^\mu \))

\[ \dot{\rho}^3(t) = 0 \]
\[ \dot{\rho}^2(t) = 2 \omega \rho^1(t) \]
\[ \dot{\rho}^1(t) = -2 \omega \rho^2(t) \]

if we multiply both side of the equation with \( \sigma_2 \)

\[ \dot{\rho}^1(t) \sigma_0 + \dot{\rho}^2(t) i \sigma_3 - \dot{\rho}^3(t) i \sigma_2 = 2 \omega \rho^1(t) \sigma_3 - 2 \omega \rho^2(t) \sigma_0 \]

taking the trace of both side

\[ \dot{\rho}^1(t) = -2 \omega \rho^2(t) \]
\[ \dot{\rho}^3(t) = 0 \Rightarrow \rho^3(t) = \rho^3 \]
\[ \dot{\rho}^2(t) = 2 \omega \rho^1(t) \]

\[ \Rightarrow \dot{\rho}^2(t) = 2 \omega \rho^1(t) \text{ and } \dot{\rho}^1(t) = -2 \omega \rho^2(t) \]

\[ \Rightarrow \rho^2(t) = A \cos(2 \omega t + \varphi) \]

Similarly \( \rho^1(t) = -A \sin(2 \omega t + \varphi) \)

for \( t = 0 , \rho = \frac{1}{2} \left[ 1 + \rho^1 \sigma_1 + \rho^2 \sigma_2 + \rho^3 \sigma_3 \right] \)

\[ \rho^i = Tr (\rho \sigma_i) \]

\[ \rho^3(t) = \rho^3 \]
\[ \rho^2(t) = \rho^2 \cos(2 \omega t) - \rho^1 \sin(2 \omega t) \]
\[ \rho^1(t) = \rho^1 \cos(2 \omega t) + \rho^2 \sin(2 \omega t) \]
since \( H = \omega \sigma_3 \) and \( \sigma_3 \) is the generator of the rotation about the \( z \) axis, then \( \rho^3(t) \) is fixed and \( \rho^1(t), \rho^2(t) \) are rotating about the \( z \) axis.

\[
\rho = \frac{1}{2} [1 + \vec{a} \sigma], \ a = |\vec{a}| \leq 1, \ \vec{a} \in \mathbb{R}^3
\]

This can be represented by sphere in \( \mathbb{R}^3 \) namely Bloch Sphere.

All pure states are on the surface of the Bloch Sphere. All mixed states are in the Bloch Sphere.

Bloch Sphere is the geometrical representation of the all possible one qubit states.

For example \( \vec{a} = (a_1, a_2, 0) \) where \( a_1^2 + a_2^2 = 1 \) defines the pure state on the \( xy \).

\[
e^{-iHt} = e^{-i\omega \sigma_z t} : \text{generate a rotation about the } z \text{ axis with the frequency } 2\omega.
\]

**SCHMIDT DECOMPOSITION**

\[
\forall |\psi > \in C^d \otimes C^d, \ |\psi > = \sum_i \sqrt{\lambda_i} |\psi_i > \otimes |\phi_i >
\]

where \( \lambda_i \geq 0 \) and \( \{|\psi_i >\} \text{ and } \{|\phi_i >\} \) are ONB.

Two ONB but, with the same indices. \( \lambda_i \)'s are called Schmidt numbers.

\[
|\psi > < \psi| = \sum_{i,j=1}^d \sqrt{\lambda_i \lambda_j} |\psi_i > < \psi_j| \otimes |\phi_i > < \phi_j|
\]

\[
\rho_A = Tr_B (|\psi > < \psi|) = \sum_i \lambda_i |\psi_i > < \psi_i| \quad \text{(trace over the basis } \{|\phi_i >\})
\]

\[
\rho_B = Tr_A (|\psi > < \psi|) = \sum_i \lambda_i |\phi_i > < \phi_i| \quad \text{(trace over the basis } \{|\psi_i >\})
\]

\[
S(\rho) = -Tr (\rho \ln \rho) = -\sum_i \rho_i \ln \rho_i
\]

\[
S(\rho_A) = S(\rho_B) = -\sum_i \lambda_i \ln \lambda_i
\]

For example \( |\psi > \in C^2 \otimes C^{100}^{100} \)

\[
|\psi > < \psi| \rightarrow \rho_A = \sum_{i=1}^2 \lambda_i |\psi_i > < \psi_i|
\]

\[
\rightarrow \rho_B = \sum_{i=1}^2 \lambda_i |\phi_i > < \phi_i|
\]

In \( \rho_B \) very small portion of the Hilbert space has Schmidt number different zero.(Incredible degeneracy of the eigenvalue 0)
The set of Schmidt numbers \( \{ \lambda_i (\psi) \} \), \( \lambda_1 (\psi) \neq 0 \) and \( \lambda_{i\neq1} (\psi) = 0 \)

If \( |\psi> \) is separable then only one of the Schmidt number is different from zero and all others are zero.

If \( |\psi>= |\chi> \otimes |\phi> \Rightarrow \lambda_i (\psi) = 0 \) but one \( |\psi>= \sum_i \sqrt{\lambda_i} |\psi_i> \otimes |\phi_i> \) ?

The state of bipartite system is separable iff its all Schmidt numbers are zero except one (and it is 1)

\[ |\psi> \in \mathbb{C}^d \otimes \mathbb{C}^d \]

\[ \rho_A = \text{Tr}_B (|\psi><\psi|) \]
\[ \rho_B = \text{Tr}_A (|\psi><\psi|) \]

\[ \rho_A = \sum_{i=1}^d \rho^i_A |\rho^i_A> <\rho^i_A| \], where \( \{|\rho^i_A> \} \) ONB in \( \mathbb{C}^d \) (for Alice)

\[ \text{Tr}_A \rho_A = \text{Tr}_A \text{Tr}_B \rho = \text{Tr} \rho = 1 \]

For Bob let \( \{|\psi^j_B> \} \) ONB in \( \mathbb{C}^d \).

\[ |\psi>= \sum_{i,j} c_{ij} |\rho^i_A> \otimes |\psi^j_B> = \sum_{i} |\rho^i_A> \otimes \left( \sum_{j} c_{ij} |\psi^j_B> \right) = \sum_{i} |\rho^i_A> \otimes |\tilde{\psi}^j_B> \]

but \( |\tilde{\psi}^j_B> = \sum_{j} c_{ij} |\psi^j_B> \) may not be orthogonalized and normalized in general.

\[ <\tilde{\psi}^i_B|\tilde{\psi}^j_B> = \sum_{j,k} c_{ik} c_{ij} <\psi^k_B|\psi^j_B> = \sum_{j} |c_{ij}|^2 \]

(from the orthogonality of \( |\psi^j_B> \)) may not be one because summing is over the only \( j \) (not \( i \))

\[ \Rightarrow |\psi>= \sum_{i} |\rho^i_A> \otimes \left| \frac{\tilde{\psi}^i_B>}{||\tilde{\psi}^i_B>||} \right| \left| \tilde{\psi}^i_B> \right| : \text{normalized} \]

\[ |\psi><\psi| = \sum_{i,j} |\rho^i_A> <\rho^j_A| \otimes |\tilde{\psi}^i_B><\tilde{\psi}^j_B> \]

The trace of this over the Bob’s degrees of freedom, gives the \( \rho_A \).

\[ \rho_A = \text{Tr}_B (|\psi><\psi|) = \text{Tr}_B \left( \sum_{i,j} |\rho^i_A> <\rho^j_A| \otimes |\tilde{\psi}^i_B><\tilde{\psi}^j_B| \right) \]

\[ \text{Tr} (|\psi><\phi|) = \sum_{i=1}^d <\chi_i|\psi><\phi|\chi_i> \text{ where } \{|\chi_i> \}_{i=1}^d \text{ ONB in } \mathbb{C}^d. \]
Choose \( |\chi_1 > = |\psi > \) and \( |\chi_i > (i = 2, 3, ..., d) \) are orthogonal
\[ \Rightarrow Tr (|\psi > < \phi |) = < \psi |\psi > < \phi |\phi > + 0 = < \phi |\psi > \]
\[ Tr (|\psi > < \phi |) = < \phi |\psi > \]
\[ \rho_A = \sum_{i,j} |\rho_A^i > < \rho_A^j | < \tilde{\psi}_B^i | \tilde{\psi}_B^j > \]
also from the beginning we have assumed that \( \rho_A = \sum_{i=1}^{d} \rho_A^i |\rho_A^i > < \rho_A^i | \)
then \( < \tilde{\psi}_B^j | \tilde{\psi}_B^i > = \delta_{ji} \rho_A^i \) i.e. orthogonal but not normalized.
It can be normalized
\[ \otimes \frac{\tilde{\psi}_B^i}{\sqrt{\rho_A^i}} > = |\phi_B^i > \Rightarrow < \phi_B^j | \phi_B^i > = \delta_{ij} \]
Now we have \( \{ |\phi_B^j > \}_{j=1}^{d} \) ONB in Bob’s \( C^d \).
\[ \Rightarrow |\psi > = \sum_{i,j} |\rho_A^i > \otimes |\phi_B^j > \sqrt{\rho_A^i} \]
Schmidt decomposition works only bipartite system (i.e. \( C^d \otimes C^d \)). For example it doesn’t work for \( C^d \otimes C^d \otimes C^d \).

Summary of the last lecture:

Unitary evolution:
\[ \rho U_t \epsilon t = U_t [\rho] = U_t \rho U_t^\dagger \]
\[ \partial_t \rho_t = -i [H, \rho_t], U_t = e^{-itH} \]
Another evolution:
\[ \rho \epsilon A [\rho] = \sum_{i} E_i \rho E_i^\dagger \text{ with } \sum_{i} E_i^\dagger E_i = 1 \]
(POVM operation.)
\[ \rho \epsilon A [\rho] = \sum_{i} P_{a_i} \rho P_{a_i} \text{ with } A = \sum_{i} a_i P_{a_i} \]
where \( P_{a_i} = |a_i > < a_i | \) (eigenprojector) (Projective measurement.)

Another (third) operation:
\[ \rho \rightarrow T [\rho] = \rho^T \]
(T stands for transpose)
For unitary evolution:
• Positive
• Trace preserving
• Pure states to pure states

For POVM operation:

• Positive
• Trace preserving
• Pure states to mixed states

For trace operation:

• Positive
• Trace preserving
• Pure states to pure states

In POVM

\[
\begin{align*}
\text{pure states} & \rightarrow \text{mixed states} \\
|\psi \rangle < \psi \rangle & \rightarrow \varepsilon \left[ |\psi \rangle < \psi \rangle \right] = \sum_i E_i |\psi \rangle < \psi \rangle E_i^\dagger
\end{align*}
\]

define \( |\phi_i \rangle = \frac{E_i |\psi \rangle}{\|E_i |\psi \rangle\|} \): normalized

\[
< \phi_i | \phi_i > = \frac{< \psi | E_i^\dagger E_i | \psi >}{< \psi | E_i^\dagger E_i | \psi >} = 1
\]

\[
\Rightarrow \varepsilon \left[ |\psi \rangle < \psi \rangle \right] = \sum_i \|E_i |\psi \rangle\| < \phi_i > < \phi_i > = \sum_i \lambda_i P_i : \text{mixture}
\]

Three of them gives from states the states. (Since all of them trace preserving and positive)

Partial transposition \( T_1 = T \otimes id : M_d (C) \otimes M_d (C) \rightarrow M_d (C) \otimes M_d (C) \) is not positive then what about \( \varepsilon_1 = \varepsilon \otimes id \) and \( U_{1t} = U_t \otimes id \) are they positive?

They are physically meaningful since they preserve positivity. (i.e. they transforms states to states) \( \varepsilon_1 \) and \( U_{1t} \) -on first system- don’t produce any unphysical changes on the second one.

**COMPLETE POSITIVITY**
Schrodinger Evolution: \[ \partial_t \psi_t = -i [H, \rho_t] \]

\[ \rho \rightarrow E_\rho [x] = Tr (\rho x) \]

\[ \downarrow \]

\[ \rho_t \rightarrow E_{\rho_t} [x] = Tr (\rho_t x) = Tr (U_t \rho_U^\dagger x) = Tr (\rho U_t^\dagger U_t x) = E_\rho [x_t] \]

(we used the cyclicity of the trace)

Heisenberg Evolution:

\[ x \in M_d(C) \rightarrow x_t = U_t^\dagger x U_t = e^{iHt} x e^{-iHt} \]

\[ \partial_t x_t = i [H, x_t] \]

Heisenberg evolution is the dual of the Schrodinger evolution. Dual of the dual is the same of the initial.
\[ \rho \rightarrow E_{\rho} [x] = Tr (\rho x) \]

\[ \varepsilon [\rho] \rightarrow E_{\varepsilon[\rho]} [x] = Tr (\varepsilon [\rho] x) = Tr \left( \sum_i E_i \rho E_i^\dagger x \right) \]

\[ = \sum_i Tr \left( E_i \rho E_i^\dagger x \right) = \sum_i Tr \left( \rho E_i^\dagger x E_i \right) = Tr \left( \rho \sum_i E_i^\dagger x E_i \right) = E_{\rho} (\varepsilon^d [x]) \]

(in \(\varepsilon^d [x]\) \(d\) stands for dual)

while

\[ \rho \rightarrow \sum_i E_i \rho E_i^\dagger, x \rightarrow \sum_i E_i^\dagger x E_i \]

\[ Tr (\varepsilon [\rho]) = Tr (\rho) : \text{trace preserving property of } \varepsilon \]

\[ \varepsilon^d [1] = 1 : \text{unital (identity preserving)} \]

Trace preserving \(\varepsilon\) iff unital \(\varepsilon^d\). \((\varepsilon^d \text{ is dual of the } \varepsilon)\)

\(\Lambda : M_d(C) \rightarrow M_d(C), \Lambda \text{ is positive (\(\Lambda\) preserves the positivity of the matrices)}\)

\(\Lambda \otimes id : M_d(C) \otimes M_d(C) \rightarrow M_d(C) \otimes M_d(C) \text{ is positive iff } \Lambda [X] = \sum_j V_j^\dagger x V_j \)

(Stinespring-Krous-Chai) \(V_j \in M_d(C)\)

These \(\Lambda\)'s are called completely positive.

- \(\Lambda\) completely positive \(\iff\) \(\Lambda \otimes id\) positive on \(M_d(C) \otimes M_N(C)\) for \(\forall N\).
- \(U_i^d [x] = U_i^\dagger x U_i\) is completely positive.
- \(\varepsilon^d [x] = \sum_i E_i^\dagger x E_i\) is completely positive.
- \(T\) is not completely positive. Because \(T \otimes id\) not preserve the positivity.
- There is no generic case characterization of positive maps (we have completely positivity characterization \(\Lambda [X] = \sum_j V_j^\dagger x V_j\) but not for positivity except \(M_2(C)\) - open problem in mathematics)

If \(\Lambda\) is only positive its form is known only for \(\Lambda : M_2(C) \rightarrow M_2(C)\) and \(M_3(C) \rightarrow M_2(C)\) (Worowowich, Störer in mid-70’s)

\(\Lambda : M_2(C) \rightarrow M_2(C)\)
Λ is positive iff Λ = Λ_{cp} + Λ_{cp}.T (cp stands for completely positive)

This property is known decomposability. For the maps Λ : M_2(C) → M_2(C) positive iff it is decomposable.

Suppose we have two POVM operation, ε_A ⊗ id_B for Alice and ε_B ⊗ id_A for Bob. Alice and Bob perform POVM operations ε_A and ε_B on their systems at the same time. Is ε_A ⊗ ε_B physical?

ε_A ⊗ ε_B [ρ_{AB}] = (ε_A ⊗ id_B)(ε_B ⊗ id_A)[ρ] = (ε_A ⊗ id_B)\(\sum_i (1_A \otimes E_i^B)\rho_{AB}(1_A \otimes E_i^B)^\dagger\) = \(\sum_{i,j} (E_j^A \otimes E_i^B)\rho_{AB}((E_j^A)^\dagger \otimes (E_i^B)^\dagger)\)

Since ε_B[ρ_B] = \(\sum_i E_i^B\rho_B(E_i^B)^\dagger\) and ε_A[ρ_A] = \(\sum_j E_j^A\rho_A(E_j^A)^\dagger\) and it is completely positive since it satisfies the criteria.

**Relations Between Positivity, Complete Positivity of Maps Λ and Entanglement of the States**

Λ is positive s.t. Λ ⊗ id preserves the positivity of separable states. Since

Λ ⊗ id \(\left[ \sum_{i,j} \lambda_{ij} P_i^A \otimes P_j^B \right] = \sum_{i,j} \lambda_{ij} [P_i^A] \otimes P_j^B, \Lambda [P_i^A] \text{ and } P_j^B \text{ are positive.}\)

Λ is completely positive is s.t. Λ ⊗ id preserves the positivity of all states (separable+entangled)

CP maps aren’t entanglement witnesses.

If Λ is CP then Λ ⊗ id is always positive. (Λ ⊗ id |ρ_{AB}| ≥ 0 where Λ is CP)

Only positive maps Λ may be entanglement witnesses.

We know that T is positive but not CP.

\(T \otimes id [[|β_{00}> < β_{00}|] = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\)
eigenvalues are \(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\) (one negative eigenvalue)

**DECOHERENCE**

Schrodinger evolution (unitary evolution) is completely positive.

\(\partial_t \rho_t = -i [H, \rho_t]\)
One example in chemical physics, molecules (with two possible state for each one) in heat bath

\[ \partial_t \rho_t = -i [H, \rho_t] + D [\rho_t] = L [\rho_t], \rho_t \in M_2 (C) \]

Here \( D [\rho_t] \) is the phenomenological description of the noise, \( D \) is a linear map on states.

- The usual request : \( \partial_t (Tr \rho_t) = 0 \), i.e. molecule cannot disappear.
  \[ \Rightarrow Tr (D [\rho_t]) = 0 \]
  (conservation of overall probability)

- \( \rho \rightarrow \gamma [\rho] = \rho_t \) : Markovian time evolution.
  \( \gamma_t = e^{tL} \) : evolution is not unitary due to the term \( D [\rho_t] \). Since process is physical then \( \gamma_t \) must be positivity preserving.

- But for quantum information theorist \( \gamma_t \) must be complete positive.

\[ \gamma_t \otimes id_2 : M_2(C) \otimes M_2(C) \rightarrow M_2(C) \otimes M_2(C) \]

**Theorem (Kosakowski-Gorini-Lindbhad)** : if we know the trace preservation, semigroup and CP then we can completely know the generator of the evolution.

\[ \gamma_t = e^{tL} \) : semigroup, since composition holds only forward in time. (\( 0 \rightarrow t_1 \rightarrow t_2 \equiv 0 \rightarrow t_2 \))

\[ \partial_t \rho_t = -i [H, \rho_t] + \sum_i V_i \rho_t V_i^\dagger - \frac{1}{2} \sum_i \left\{ V_i^\dagger V_i, \rho_t \right\} \]

On the right hand side, the first term is unitary part, the second term is noise, the third term is damping (friction) term.

Trace is conserved

\[ Tr (\partial_t \rho_t) = \partial_t Tr (\rho_t) = Tr (-i [H, \rho_t]) + Tr \left( \sum_i V_i \rho_t V_i^\dagger \right) - Tr \left( \frac{1}{2} \sum_i \left\{ V_i^\dagger V_i, \rho_t \right\} \right) \]

with the help of the cyclic property of the trace we can find

\[ Tr (\partial_t \rho_t) = \partial_t Tr (\rho_t) = 0 \]

i.e. trace is preserved.

**Noise Term** : it is a kind of POVM which gives from pure states to mixed states. It is increase the entropy.
The equation is also description of the qubit in the noisy channel. This gives
the evolution of the qubit state in the noisy channel.

Suppose that \( V_i = V_i^\dagger \) and for initial \( \rho \), \( [\rho, V_i] = 0 \) then second and the third
terms in the right hand side of the equation cancels each other. This means that
state evolves unitary in the noisy channel.

\[
\rho = \begin{pmatrix} x & y \\ y & 1-x \end{pmatrix} \rightarrow \rho_t = \begin{pmatrix} x_t & y_t \\ y_t & 1-x_t \end{pmatrix}
\]

define \( T_y \): the lifetime of the off diagonal elements and \( T_x \): the lifetime of the
diagonal elements. If the description is given by the equation then \( T_y \leq T_x^2 \).

CP avoid the some illogical effects. (Entanglement with something far away example which is uncontrollable)

S+R closed. (system+reservoir)

\( U_t \rho_{SR} U_t^\dagger \): evolution of the whole -closed- system.

\( Tr_R \left( U_t \rho_{SR} U_t^\dagger \right) = \rho_S (t) \): this is difficult operation since \( U_t \) is a huge matrix.
(It contains all interaction and degrees of freedom is infinitely many)

\[ \rho_{SR} = \rho_S \otimes \rho_R, \rho_S \Gamma_t \rho_S (t), \text{where } \Gamma_t \text{ is CP (under the condition } \rho_{SR} = \rho_S \otimes \rho_R \) \]

Since \( \Gamma_t \) is not (????) a semigroup then \( \Gamma_t \Gamma_s \neq \Gamma_{t+s} \)

\( \Gamma_t \) is huge operator because it has infinitely many degrees of freedom. With
Markov approximation \( \Gamma_t \rightarrow \gamma_t, (\gamma_t \gamma_s = \gamma_{t+s} \) it holds the composition) Since \( \gamma_t \) is
semigroup then \( \Gamma_t \) becomes approximated semigroup.

CP can defines the noise CP is not a witness of entanglement.

Horodecki's Theorem : \( \rho \in M_2(C) \otimes M_2(C) \) is separable iff \( \Lambda \otimes id [\rho] \geq 0 \)

PPT \( \rho \)'s are in \( M_2(C) \) are separable and vice versa.

In \( M_d(C) \) \( (d \geq 3) \) there PPT states which are entangled.

Theorem : \( \rho \in M_d(C) \otimes M_d(C) \) is separable iff \( \Lambda \otimes id [\rho] \geq 0 \) for all positive
\( \Lambda \). (The way \( \rightarrow \) was proven but it is hard to prove the reverse way)

Corollary : \( \rho \in M_d(C) \otimes M_d(C) \) is separable iff \( T \otimes id [\rho] \geq 0 \)

\( \Lambda \) is positive \( \Rightarrow \Lambda = \Lambda^1_{cp} + \Lambda^2_{cp}.T \)

\( \rightarrow \) if \( \rho \) is separable then \( T \otimes id [\rho] \geq 0 \)
\[(\Lambda_1 \otimes \text{id}) [\rho] + (\Lambda_2 \otimes \text{id}) (T \otimes \text{id}) [\rho] = (\Lambda_1 + \Lambda_2 T) \otimes \text{id}) [\rho] = \Lambda [\rho]\]

with performing summation

\[(\Lambda_1 \otimes \text{id}) [\rho] + (\Lambda_2 \otimes \text{id}) (T \otimes \text{id}) [\rho] = (\Lambda_1 + \Lambda_2 T) \otimes \text{id}) [\rho] = \Lambda [\rho]\]

Then, if the partial transposition preserves the positivity then all decomposable \(\Lambda\) will preserve the positivity.

**Theorem:** \(\rho \in M_d(C) \otimes M_d(C)\) is separable iff \(\text{Tr} (\rho \omega) \geq 0\) for all \(\omega \in M_d(C) \otimes M_d(C)\) s.t.

\(\text{Tr} (\omega P \otimes Q) \geq 0\) for all \(P = |\psi><\psi|\) and \(Q = |\phi><\phi|\) where \(|\psi>, |\phi> \in M_d(C)\).

There is a correspondence between the \(\omega\) matrices and the \(\Lambda\) maps (in the theorem before the last corollary)

\(\omega \in M_d(C) \otimes M_d(C) \rightarrow \Lambda_\omega : \quad \text{positive map on } M_d(C)\)

\(\Lambda\) positive on \(M_d(C) \rightarrow \omega_\Lambda \in M_d(C) \otimes M_d(C)\) satisfying last theorem.

\(\omega \in M_d(C) \otimes M_d(C) \rightarrow \Lambda_\omega : \quad \text{positive map on } M_d(C)\)

\(\text{Tr} (\omega P \otimes Q) \geq 0\) for all \(P = |\psi><\psi|\) and \(Q = |\phi><\phi|\).

\(\Lambda : M_d(C) \rightarrow M_d(C), X \in M_d(C)\)

\[X = \sum_{i,j}^d X_{ij}|i><j|\]

\(|i><j|\) are called matrix units. \(|i>\}_{i=1}^d \text{ONB in } C^d\).

\[\Lambda [X] = \sum_{i,j}^d X_{ij} \Lambda [i><j]|\]

here again \(\Lambda [i><j]|\) is matrix. With the completeness relation,

\[\Lambda [X] = \sum_{i,j,p,q} X_{ij} <p|\Lambda [i><j]|q > |q><q|\]

here \(<p|\Lambda [i><j]|q >\) is matrix in \(M_d(C) \otimes M_d(C)\). It has \(d^4\) elements

\[\Lambda \rightarrow X_{\Lambda} = \sum_{i,j,p,q} <p|X_{\Lambda}|qj > |p><q| \otimes |i><j|\]

here \(<p|X_{\Lambda}|qj >\) is the \(<p|\Lambda [i><j]|q >\) entries of the matrix
Whenever $\Lambda$ is given then one can construct the corresponding matrix $X_{\Lambda}$ in this way.

When matrix $X$ is given,

$$X = \sum_{i,j,p,q} <pi|X_{\Lambda}|qj> |p><q| \otimes |i><j| \rightarrow \Lambda_X = \{ <pi|X_{\Lambda}|qj> \}$$

Example: flip operator. $V : C^d \otimes C^d \rightarrow C^d \otimes C^d$

$V|\psi \otimes \phi> = |\phi \otimes \psi>$, $V = V^\dagger$, $V^2 = 1$ (eigenvalues are $\pm 1$ i.e. not positive matrix) $\psi$ is in the first Hilbert space and $\phi$ is in the second Hilbert space. find the matrix $\Lambda_V$ which is associated with flip operator ($V$).

$$\Lambda_V = <p|\Lambda_V |i><j| |q> = <pi|V|qj> = <pi|V^\dagger|jq> = \delta_{pj}\delta_{iq}$$

$$\Lambda_V = \sum_{i,j} X_{ij} \Lambda_V |i><j| = \sum_{i,j,p,q} X_{ij} <p|\Lambda_V |i><j| |q> |q><q|$$

$$\Lambda_V = \sum_{i,j} X_{ij} |j><i| = Tr(X)$$

$\Lambda_V$ associated with the flip operator ($V$) is transposition.

$$\omega = \sum_{i,j,p,q} <pi|\omega|qj> |p><q| \otimes |i><j| \rightarrow \Lambda_\omega = \{ <p|\Lambda_\omega |i><j| |q> \}$$

$$Tr(\omega P \otimes Q) \geq 0 \text{ for all } P = |\psi><\psi| \text{ and } Q = |\phi><\phi|$$

$$Tr(\omega P \otimes Q) = Tr(\omega|\psi><\psi| \otimes |\phi><\phi|) = <\psi \otimes \phi|\omega|\psi \otimes \phi> = \sum_{p,i,q,j} \psi_i^* \psi_j <pi|\omega|qj> \psi_q \phi_j$$

here $<pi|\omega|qj>$ is the matrix elements of $\omega$.

$$Tr(\omega P \otimes Q) = \sum_{p,i,q,j} \psi_i^* \psi_j <pi|\Lambda |i><j| |q> \psi_q \phi_j$$

$$= <\psi| \left( \sum_{i,j} \phi_i^* \phi_j \Lambda_\omega |i><j| \right) |\psi> = <\psi|\Lambda_\omega |\phi^*><\phi^*| |\psi> \geq 0 (\forall \psi, \phi)$$

$$\Rightarrow \Lambda_\omega |\phi^*><\phi^*| \geq 0$$

(since $Tr(\omega P \otimes Q) \geq 0$)
If $A = A^\dagger \geq 0 \Rightarrow A = \sum_i a_i |a_i><a_i|$

$\Lambda_{\omega} [||\phi^*><\phi^*||] \geq 0$ is general result (i.e. it exist for any positive matrix $A$)

*Theorem 0*: $\rho$ is separable iff $\text{Tr} (\rho \omega) \geq 0, \forall \omega$ s.t. $\text{Tr} (\omega P \otimes Q) \geq 0, \forall P, Q$ projectors.

*Theorem 1*: $\rho$ is separable iff $\Lambda \otimes \text{id} [|\rho >] \geq 0, \forall$ positive $\Lambda$ on $M_d(C)$

The connection between the $\Lambda$ and the $\omega$ established above.

$M_d(C) \otimes M_d(C)$ are matrix on $C^d \otimes C^d, |\psi_+ > \in C^d \otimes C^d$ (namely totally symmetric vector)

$$|\psi_+ > = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |ii >$$

(weights of the $|ii >$ are the same)

Given $\Lambda : M_d(C) \to M_d(C), \Lambda \otimes \text{id} [|\psi_+><\psi_+|] = \frac{1}{d} \Lambda \otimes \text{id}$

$$\Lambda \otimes \text{id} [|\psi_+><\psi_+|] = \frac{1}{d} \Lambda \otimes \text{id} \left[ \sum_{i,j} |ii><jj| \right]$$

$$= \frac{1}{d} \Lambda \otimes \text{id} \left[ \sum_{i,j} |i><j| \otimes |i><j| \right]$$

$$= \frac{1}{d} \sum_{i,j} \Lambda [|i><j|] \otimes [|i><j|]$$

$$= \frac{1}{d} \sum_{i,j} \sum_{p,q} <p|\Lambda [|i><j|] |q > |p><q| \otimes [|i><j|]$$

$$= \frac{1}{d} X_\Lambda$$

When we have given $\Lambda$ and we want to associate one matrix to it (finding corresponding matrix) we only do construct $\Lambda \otimes \text{id}$ and lets act on $|\psi_+><\psi_+|$. 

$$\frac{1}{d} X_\Lambda = \Lambda \otimes \text{id} [|\psi_+><\psi_+|]$$

$M_d(C) \otimes M_d(C) : \delta$ space of density matrices $\supset \delta_{\text{sep}} = \left\{ \sum_{i,j} \lambda_{ij} P_i^1 \otimes P_j^2 \right\}$

characteristics of $\delta_{\text{sep}}$

- $\delta_{\text{sep}}$ is a convex set. $\rho_1, \rho_2 \in \delta_{\text{sep}}$ then $t \rho_1 + (1-t) \rho_2 \in \delta_{\text{sep}}, \forall t \in [0,1]$ (i.e. convex linear combination of $\rho_1, \rho_2$ is again in the $\delta_{\text{sep}}$)

- $\delta_{\text{sep}}$ is closed. Topology is given by Hilbert-Schmidt norm.
Hilbert-Schmidt scalar product: \( A, B \in M_d(C) \mapsto (A, B) = \text{Tr} (A^\dagger B) \)

And, with this product one can induce the norm:

\[
\|A\|_{HS}^2 = \text{Tr} (A^\dagger A) = (A, A)
\]

(HS stands for Hilbert-Schmidt)

\[d(\rho_{\text{ent}}, \delta_{\text{sep}}) = \inf_{\rho' \in \delta_{\text{sep}}} \|\rho_{\text{ent}} - \rho'\|_{HS} = \|\rho_{\text{ent}} - \rho^*\|_{HS}\]

(* stands for separable)

\[\|\rho_{\text{ent}} - \rho^*\|_{HS}^2 \geq 0 \text{ with another separable } \rho' \text{ (and since } \|\rho_{\text{ent}} - \rho^*\|_{HS}^2 \text{ is minimal)}\]

\[\|\rho_{\text{ent}} - \rho^*\|_{HS}^2 \leq \|\rho_{\text{ent}} - ((1 - t) \rho^* + t\rho')\|_{HS}^2\]

here \(((1 - t) \rho^* + t\rho')\) is again in the \(\delta_{\text{sep}}\) since it is linear convex combination.

\[\Rightarrow 0 \leq \|\rho_{\text{ent}} - \rho^*\|_{HS}^2 \leq \|\rho_{\text{ent}} - \rho^* + t (\rho^* - \rho')\|_{HS}^2 \leq \|\rho_{\text{ent}} - (1 - t) \rho^* + t\rho')\|_{HS}^2 = t^2 (\rho^* - \rho') + 2t (\rho_{\text{ent}} - \rho^*, \rho^* - \rho')\]

\[0 \leq t \Rightarrow 0 \leq (\rho_{\text{ent}} - \rho^*, \rho^* - \rho')\]

\[0 \leq \|\rho_{\text{ent}} - \rho^*\|_{HS}^2 + 2t (\rho_{\text{ent}} - \rho^*, \rho^* - \rho')\]

\[0 \leq \|\rho_{\text{ent}} - \rho^*\|_{HS}^2 + \|\rho_{\text{ent}} - \rho^* + \rho_{\text{ent}} - \rho')\|_{HS}^2\]

define \(-N = \rho_{\text{ent}} - \rho^*\) (it is not density matrix, but still matrix)

\[\Rightarrow 0 \leq -\|N\|_{HS}^2 - (N, \rho_{\text{ent}}) + (N, \rho') \geq (N, \rho') \geq \|N\|_{HS}^2 + (N, \rho_{\text{ent}})\]

Now, from \(-N = \rho_{\text{ent}} - \rho^*\) we can say that \(\|N\|_{HS}^2 > 0\) since \(\rho_{\text{ent}}\) out of the set \(\delta_{\text{sep}}\), which have \(\rho^*\).

Then

\[(N, \rho') \geq (N, \rho_{\text{ent}})\]

taking the infimum of \(\rho' \in \delta_{\text{sep}}\)

\[\inf_{\rho' \in \delta_{\text{sep}}} (N, \rho') = \alpha \Rightarrow (N, \rho') \geq \alpha \geq (N, \rho_{\text{ent}})\]

\[\forall \rho' \in \delta_{\text{sep}}, (N, \rho') \geq \alpha \geq (N, \rho_{\text{ent}}) \quad \text{where} \quad N = \rho^* - \rho_{\text{ent}}\]
This means that,
\[
\text{Tr} (N\rho') \geq \alpha \geq \text{Tr} (N\rho_{\text{ent}})
\]
⇒ \(-\alpha + \text{Tr} (N\rho') \geq 0 \geq \text{Tr} (N\rho_{\text{ent}}) - \alpha\)

Since \(\text{Tr} (\rho') = \text{Tr} (\rho_{\text{ent}}) = 1\) then,
\[
-\alpha \text{Tr} (\rho') + \text{Tr} (N\rho') \geq 0 \geq \text{Tr} (N\rho_{\text{ent}}) - \alpha \text{Tr} (\rho_{\text{ent}})
\]
\[
(N - \alpha)\rho' \geq 0 \geq (N - \alpha)\rho_{\text{ent}}
\]

\(N - \alpha = \omega\)
⇒ \(\text{Tr} (\omega\rho') \geq 0 \geq \text{Tr} (\omega\rho_{\text{ent}})\)

Mean value of matrix \(\omega\) with respect to state \(\rho_{\text{ent}}\) is negative and with respect to the state \(\rho'\) (which is separable state) is positive.

**Theorem 0**: \(\rho\) is separable iff \(\text{Tr} (\rho\omega) \geq 0, \forall \omega\) s.t. \(\text{Tr} (\omega P \otimes Q) \geq 0, \forall P, Q\) projectors.

**Proof**: 

\((\rightarrow)\)

\(\rho\) is separable ⇒ \(\rho = \sum_{i,j} \lambda_{ij} P^1_i \otimes P^2_j\)
⇒ \(\text{Tr} (\rho) = \text{Tr} \left( \omega \sum_{i,j} \lambda_{ij} P^1_i \otimes P^2_j \right)\)
= \(\sum_{i,j} \lambda_{ij} \text{Tr} (\omega P^1_i \otimes P^2_j)\)

Since \(\omega\) is a matrix s.t. (by definition in theorem) \(\text{Tr} (\omega P \otimes Q) \geq 0\) i.e. \(\rho\) is separable⇒ \(\text{Tr} (\omega P^1_i \otimes P^2_j)\)

\((\leftarrow)\)

\(\text{Tr} (\rho\omega) \geq 0, \forall \omega\) s.t. \(\text{Tr} (\omega P \otimes Q) \geq 0, \forall P, Q\) projectors.

Suppose \(\rho\) is entangled ⇒ \(\text{Tr} (\rho\omega) < 0\) by the result getting above (\(\text{Tr} (\omega\rho') \geq 0 \geq \text{Tr} (\omega\rho_{\text{ent}})\)). But this is contradiction then \(\rho\) is separable.

For proof of the Theorem 1, use Theorem 0 and the result which is found above (\(1/2 X_\Lambda = \Lambda \otimes id [|\psi+><\psi+|]\))

**Theorem 1**: \(\rho\) is separable iff \(\Lambda \otimes id [\rho] \geq 0, \forall\) positive \(\Lambda\) on \(M_d (C), \rho \in M_d (C) \otimes M_d (C)\)

\((\rightarrow)\)

this part was proved.
if $\Lambda \otimes \text{id} [\rho] \geq 0 \Rightarrow \rho$ is separable for all positive $\Lambda$.

$\Lambda \otimes \text{id} [\rho] \geq 0$ then $Tr (\Lambda \otimes \text{id} [\psi_+ > < \psi_+])$ for $\forall$ positive $\Lambda$

$Tr (\Lambda \otimes \text{id} [\psi_+ > < \psi_+]) = Tr (\rho \Lambda^d \otimes \text{id} [\psi_+ > < \psi_+]) = Tr (\rho \omega)$

with using duality shifting. $\frac{\omega}{d} \in M_d (C) \otimes M_d (C)$ where $\omega$ is the corresponding matrix of the operator $\Lambda$. $\omega$ s.t. $Tr (\omega P \otimes Q) \geq 0$ if $\Lambda$ is positive $\Leftrightarrow \Lambda^d$ is positive.

We arrive :

if $\forall$ positive $\Lambda$, $\Lambda \otimes \text{id} [\rho] \geq 0 \Rightarrow Tr (\omega \Lambda P \otimes Q) \geq 0$, for $\forall P, Q$ projectors.

Theorem 0 says $(Tr (\omega \Lambda P \otimes Q) \geq 0)$ $\rho$ is separable.

How can one construct in $d \geq 3$ PPT and entangled states?

Carotherdory Theorem : $\rho \in \delta_{sep}$ for $M_d (C) \otimes M_d (C)$ can be written as a convex combination of at most $d^2$ projectors of the form $P_i^1 \otimes P_j^2 = |\psi_1^i > < \psi_1^i| \otimes |\psi_2^j > < \psi_2^j|$. These $d^2$ projectors are called extreme generator state. Any $\rho \in \delta_{sep}$, $\rho = \sum_{ij} \lambda_{ij} |\psi_1^i > < \psi_1^i| \otimes |\psi_2^j > < \psi_2^j|$, $d^2$ projectors can generate all separable states.

Range of Operator :

$Range (\rho) \equiv \{ |\psi > \in C^d \otimes C^d \text{ s.t. } \exists |\phi > \in C^d \otimes C^d : \rho |\phi > = |\psi > \}$

Example :

$\rho = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, \(Range (\rho) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\)

$\rho^{T_1} = \sum_{ij} \lambda_{ij} |\psi_1^{1*} > < \psi_1^{1*}| \otimes |\psi_2^{2*} > < \psi_2^{2*}|$ : partial transposition on $1$

$\{ |\psi_1^* > \otimes |\psi_2^* > \}$ spanning the range of $\rho$ and s.t.

$\{ |\psi_1^* > \otimes |\psi_2^* > \}$ spanning the range of $\rho^{T_1}$.

$\rho |\psi > = \sum_{ij} \lambda_{ij} |\psi_1^i > < \psi_1^i| \otimes |\psi_2^j > < \psi_2^j| |\psi >$

$= \sum_{ij} \lambda_{ij} < \psi_1^i \otimes \psi_2^j |\psi > |\psi_1^i > \otimes |\psi_2^j >$ : range of $\rho$.

(here $< \psi_1^i \otimes \psi_2^j |\psi > |\psi_1^i > \otimes |\psi_2^j >$ is a vector)
The range is generated (spanned) by $|\psi_1^1 > \otimes |\psi_2^2 >$. In a similar way one can prove the second one.

If $\text{Range} (\rho)$ cannot be linearly spanned by product vectors can $\rho$ be separable? From theorem 2 NO!

If we can find $\rho$ s.t. it’s range is cannot be spanned by $\{|\psi_1^1 > \otimes |\psi_2^2 >\}$ then it is entangled. (From theorem 2)

$M_3(C) \otimes M_3(C)$, Hilbert space is $C^3 \otimes C^3$, $\{|0>, |1>, |2>\}$ ONB for the two of $C^3$. Choose

$$
|v_1 > = |0 > \otimes \frac{1}{\sqrt{2}} (|0 > - |1 >)
|v_2 > = \frac{1}{\sqrt{2}} (|0 > - |1 >) \otimes |2 >
|v_3 > = |2 > \otimes \frac{1}{\sqrt{2}} (|1 > - |2 >)
|v_4 > = \frac{1}{\sqrt{2}} (|1 > - |2 >) \otimes |0 >
|v_5 > = \frac{1}{\sqrt{3}} (|0 > + |1 > + |2 >) \otimes \frac{1}{\sqrt{3}} (|0 > + |1 > + |2 >)
$$

These are five orthonormal vectors. $\{|v_i >\}_{i=1}^5$ generate $K \subset C^9$ and the dimension of $K$ is 5. $C^9 \setminus K = K^\perp$ and $\dim (K^\perp) = 4$.

There doesn’t exist complete of this basis set like $|\phi_1 > \otimes |\phi_2 >$. It couldn’t orthogonal all the $\{|v_i >\}_{i=1}^9$.

$P : C^9 \rightarrow K$, where $K$ is subspace spanned by $\{|v_i >\}_{i=1}^5$, and $P$ is the projection.

$$
\rho = \frac{1-P}{4} \ (4 \ is \ about \ the \ normalization \ s.t. \ Tr(\rho) = 1, \ Tr(1) = 9, \ P = \sum_{i=1}^5 |v_i > < v_i|, \ Tr(P) = 5)
$$

Is $\rho$ positive?

$P$ is projection $P^2 = P$ and $(1 - P)^2 = (1 - P) \Rightarrow \rho$ is positive.

$\rho$ is entangled since $\text{Range} (\rho) = K^\perp$ which cannot be spanned by product vectors.

$\rho$ is entangled, but is it PPT?

$$
|v_i > = \{|\psi_{a_i}^1 > \otimes |\psi_{b_i}^2 >\}
$$

$$
\rho = \frac{1}{4} - \frac{1}{4} \sum_{ab} |\psi_{a}^{1*} > < \psi_{a}^{1*} | \otimes |\psi_{b}^{2*} > < \psi_{b}^{2*} | : \ \rho^{T_1} \ remains \ positive.
$$

(here $\sum_{ab} |\psi_{a}^{1*} > < \psi_{a}^{1*} | \otimes |\psi_{b}^{2*} > < \psi_{b}^{2*} |$ is also projector)
Then the state is PPT and entangled.