### Introduction

In this chapter you work with production functions, relating output of a firm to the inputs it uses. This theory will look familiar to you, because it closely parallels the theory of utility functions. In utility theory, an indifference curve is a locus of commodity bundles, all of which give a consumer the same utility. In production theory, an isoquant is a locus of input combinations, all of which give the same output. In consumer theory, you found that the slope of an indifference curve at the bundle \((x_1, x_2)\) is the ratio of marginal utilities, \(MU(x_1, x_2)/MU(x_1, x_2)\). In production theory, the slope of an isoquant at the input combination \((x_1, x_2)\) is the ratio of the marginal products, \(MP(x_1, x_2)/MP(x_1, x_2)\). Most of the functions that we gave as examples of utility functions can also be used as examples of production functions.

There is one important difference between production functions and utility functions. Remember that utility functions were only “unique up to monotonic transformations.” In contrast, two different production functions that are monotonic transformations of each other describe different technologies.

#### Example

If the utility function \(U(x_1, x_2) = x_1 \times x_2\) represents a person’s preferences, then so would the utility function \(U’(x_1, x_2) = (x_1 + x_2)^2\). A person who had the utility function \(U(x_1, x_2)\) would have the same indifference curves as a person with the utility function \(U’(x_1, x_2)\) and would make the same choices from every budget. But suppose that one firm has the production function \(f(x_1, x_2) = x_1 \times x_2\) and another has the production function \(f’(x_1, x_2) = (x_1 + x_2)^2\). It is true that the two firms will have the same isoquants, but they certainly do not have the same technology. If both firms have the input combination \((x_1, x_2) = (1, 1)\), then the first firm will have an output of 2 and the second firm will have an output of 4.

Now we investigate “returns to scale.” Here we are concerned with the change in output if the amount of every input is multiplied by a number \(t > 1\). If multiplying inputs by \(t\) multiplies output by more than \(t\), then there are increasing returns to scale. If output is multiplied by exactly \(t\), there are constant returns to scale. If output is multiplied by less than \(t\), then there are decreasing returns to scale.

#### Example

Consider the production function 
\[
\begin{align*}
    f(x_1, x_2) & = x_1^2 + x_2^2 & \text{for } t 
\end{align*}
\]

If we multiply the amount of each input by \(t\), then output will be 
\[
\begin{align*}
    f(tx_1, tx_2) & = (tx_1)^2 + (tx_2)^2 = t^2x_1^2 + t^2x_2^2 
\end{align*}
\]

To compare \(f(x_1, x_2)\) to \(f(tx_1, tx_2)\), factor out the expressions involving \(t\) from the last equation. You get 
\[
\begin{align*}
    f(x_1, x_2) & = t^2(f(x_1, x_2)) \quad \text{for } t > 2
\end{align*}
\]

Therefore when all inputs are multiplied by \(t\), output is also multiplied by \(t\). It follows that this production function has constant returns to scale.

You will also need to determine whether the marginal product of each single factor of production increases or decreases as you increase the amount of that factor without changing the amount of other factors. Those of you who know calculus will recognize that the marginal product of a factor is the first derivative of output with respect to the amount of that factor. Therefore the marginal product of a factor will decrease, increase, or stay constant as the amount of the factor increases depending on whether the second derivative of the production function with respect to the amount of that factor is negative, positive, or zero.

#### Example

Consider the production function \(f(x_1, x_2) = x_1^4 x_2^2\) . The marginal product of factor \(1\) is \(4x_1^3 x_2^2\). This is a decreasing function of \(x_1\), as you can verify by taking the derivatives of the marginal products with respect to \(x_1\). Similarly, you can show that the marginal product of \(x_2\) decreases as \(x_2\) increases.

### Marginal Products and Technical Rates of Substitution

<table>
<thead>
<tr>
<th>(f(x_1, x_2))</th>
<th>(MP(x_1, x_2))</th>
<th>(MP(x_1, x_2))</th>
<th>(TRS(x_1, x_2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x_1 \times x_2)</td>
<td>1</td>
<td>2</td>
<td>(-1/2)</td>
</tr>
<tr>
<td>(ax_1 + bx_2)</td>
<td>a</td>
<td>b</td>
<td>(-a/b)</td>
</tr>
<tr>
<td>(5x_1 x_2)</td>
<td>50x_2</td>
<td>50x_1</td>
<td>(-25)</td>
</tr>
<tr>
<td>((x_1 + x_2)^2)</td>
<td>(2x_1 + 2x_2)</td>
<td>(2x_1 + 2x_2)</td>
<td>(-1)</td>
</tr>
<tr>
<td>((x_1 + x_2)^\alpha)</td>
<td>((x_1 + x_2)^\alpha)</td>
<td>((x_1 + x_2)^\alpha)</td>
<td>(-\alpha/2)</td>
</tr>
<tr>
<td>(x_1 + (t+1)x_2)</td>
<td>((t+1)x_2)</td>
<td>((t+1)x_2)</td>
<td>(-\frac{(t+1)}{t})</td>
</tr>
<tr>
<td>((x_1 + a)x_2 + b)</td>
<td>(x_1 + a)</td>
<td>(x_2 + b)</td>
<td>(-\frac{t}{t+1})</td>
</tr>
<tr>
<td>(x_1 + c)</td>
<td>(a)</td>
<td>(a)</td>
<td>(\frac{t}{t+1})</td>
</tr>
<tr>
<td>(x_1 + x_2)</td>
<td>(x_1 + x_2)</td>
<td>(x_1 + x_2)</td>
<td>(-\frac{t}{t+1})</td>
</tr>
<tr>
<td>((x_1 + x_2)^\beta)</td>
<td>((x_1 + x_2)^\beta)</td>
<td>((x_1 + x_2)^\beta)</td>
<td>(-\frac{t}{t+1})</td>
</tr>
</tbody>
</table>

### Returns to Scale and Changes in Marginal Products

For each production function in the table below, put an \(L\), \(C\), or \(D\) in the first column if the production function has increasing, constant, or decreasing returns to scale. Put an \(L\), \(C\), or \(D\) in the second (third) column, depending on whether the marginal product of factor 1 (factor 2) is increasing, constant, or decreasing, as the amount of that factor alone is varied.

<table>
<thead>
<tr>
<th>(f(x_1, x_2))</th>
<th>Scale</th>
<th>(MP_1)</th>
<th>(MP_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x_1 \times x_2)</td>
<td>(C)</td>
<td>(C)</td>
<td>(C)</td>
</tr>
<tr>
<td>((x_1 + x_2)^\alpha)</td>
<td>(D)</td>
<td>(D)</td>
<td>(D)</td>
</tr>
<tr>
<td>((x_1 + a)x_2 + b)</td>
<td>(I)</td>
<td>(C)</td>
<td>(I)</td>
</tr>
<tr>
<td>((x_1 + x_2)^\beta)</td>
<td>(C)</td>
<td>(D)</td>
<td>(D)</td>
</tr>
<tr>
<td>((x_1 + x_2)^\gamma)</td>
<td>(D)</td>
<td>(D)</td>
<td>(D)</td>
</tr>
<tr>
<td>((x_1 + x_2)^\delta)</td>
<td>(C)</td>
<td>(D)</td>
<td>(D)</td>
</tr>
</tbody>
</table>

(c) On the graph below, plot some input combinations that give her an output of 4 bushels. Sketch a production isocost that runs through those points. The points on the isocost that give her an output of 4 bushels all satisfy the equation \(T = 16/L\).
This technology demonstrates (increasing, constant, decreasing) returns to scale. Constant returns to scale.

In the short run, Prunella cannot vary the amount of land she uses. On the graph below, use blue ink to draw a curve showing Prunella’s output as a function of labor input if she has 1 unit of land. Locate the points on your graph at which the amount of labor is 0, 1, 4, 9, and 16 and label them. The slope of this curve is known as the marginal product of labor. Is this curve getting steeper or flatter as the amount of labor increases? Flatter.

There are (constant, increasing, decreasing) returns to scale.

The marginal product of factor 2 is $3/2x_1^{1/2}x_2^{1/2}$ and it (increases, remains constant, decreases) for small increases in $x_2$.

An increase in the amount of $x_2$ (increases, leaves unchanged, decreases) the marginal product of $x_1$.

The technical rate of substitution between $x_2$ and $x_1$ is $-x_2/3x_1$.

Does this technology have diminishing technical rate of substitution? Yes.

This technology demonstrates (increasing, constant, decreasing) increasing returns to scale.

The production function exhibits (constant, increasing, decreasing) returns to scale.

In the short run, capital is fixed at 4 units. Labor is variable. On the graph below, use blue ink to draw output as a function of labor input in the short run. Use red ink to draw the marginal product of labor as a function of labor input in the short run. The average product of labor defined as total output divided by the amount of labor input. Use black ink to draw the average product of labor as a function of labor input in the short run.

General Monsters Corporation has two plants for producing juggernauts, one in Flint and one in Inkster. The Flint plant produces according to $f(x_1, x_2) = \min(x_1, 2x_1)$ and the Inkster plant produces according to $f(x_1, x_2) = \min(2x_1, x_2)$, where $x_1$ and $x_2$ are the inputs.

On the graph below, use blue ink to draw the isoquant for producing 40 juggernauts at the Flint plant. Use red ink to draw the isoquant for producing 40 juggernauts at the Inkster plant.
(b) Suppose that the firm wishes to produce 20 juggernauts at each plant. How much of each input will the firm need to produce 20 juggernauts at the Flint plant? $x_1 = 20, x_2 = 10$. How much of each input will the firm need to produce 20 juggernauts at the Inkster plant? $x_1 = 10, x_2 = 20$. Label with an $a$ on the graph, the point representing the total amount of each of the two inputs that the firm needs to produce a total of 40 juggernauts, 20 at the Flint plant and 20 at the Inkster plant.

(c) Label with a $b$ on your graph the point that shows how much of each of the two inputs is needed in total if the firm is to produce 10 juggernauts in the Flint plant and 30 juggernauts in the Inkster plant. Label with an $c$ the point that shows how much of each of the two inputs that the firm needs in total if it is to produce 30 juggernauts in the Flint plant and 10 juggernauts in the Inkster plant. Use a black pen to draw the firm’s isoquant for producing 40 units of output if it can split production in any manner between the two plants. Is the technology available to the firm convex? Yes.

18.6 (0) You manage a crew of 160 workers who could be assigned to make either of two products. Product A requires 2 workers per unit of output. Product B requires 4 workers per unit of output. Write an equation to express the combinations of products A and B that could be produced using exactly 160 workers. Suppose that the production function is $B = 18.9 (0)$ 160. How much of each input will the firm need to produce 20 juggernauts at the Inkster plant? $x_1 = 10, x_2 = 20$. Label with an $a$ on the graph, the point representing the total amount of each of the two inputs that the firm needs to produce a total of 40 juggernauts, 20 at the Flint plant and 20 at the Inkster plant.

(c) Label with a $b$ on your graph the point that shows how much of each of the two inputs is needed in total if the firm is to produce 10 juggernauts in the Flint plant and 30 juggernauts in the Inkster plant. Label with a $c$ the point that shows how much of each of the two inputs that the firm needs in total if it is to produce 30 juggernauts in the Flint plant and 10 juggernauts in the Inkster plant. Use a black pen to draw the firm’s isoquant for producing 40 units of output if it can split production in any manner between the two plants. Is the technology available to the firm convex? Yes.

18.7 (0) A firm has the production function $f(x,y) = \min\{2x, 2y\}$. On the graph below, use red ink to sketch a couple of production isoquants for this firm. A second firm has the production function $f(x,y) = x + \min\{x, y\}$. Do either or both of these firms have constant returns to scale? Do either or both of these firms have constant returns to scale? For all positive values.

18.10 (0) Suppose that the production function is $f(x_1, x_2) = (x_1^a + x_2^a)^{1/a}$, where $a$ and $b$ are positive constants.

(a) For what positive values of $a$ and $b$ are there decreasing returns to scale? $ab < 1$. Constant returns to scale? $ab = 1$. Increasing returns to scale? $ab > 1$.

18.11 (0) Suppose that a firm has the production function $f(x_1, x_2) = \sqrt{x_1} + x_2^b$.

(a) The marginal product of factor 1 (increases, decreases, stays constant) decreases as the amount of factor 1 decreases. The marginal product of factor 2 (increases, decreases, stays constant) increases as the amount of factor 2 increases.

(b) This production function does not satisfy the definition of increasing returns to scale, constant returns to scale, or decreasing returns to scale. How can this be? Returns to scale are different depending on the ratio in which the factors are used. Find a combination of inputs such that doubling the amount of both inputs will more than double the amount of output. $x_1 = 4, x_2 = 4$, for example. Find a combination of inputs such that doubling the amount of both inputs will less than double the amount of output. $x_1 = 1, x_2 = 0$, for example.