CHAPTER 11

VECTOR MECHANICS FOR ENGINEERS: DYNAMICS

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Kinematics of Particles
Contents

Introduction

Rectilinear Motion: Position, Velocity & Acceleration

Determination of the Motion of a Particle

Sample Problem 11.2

Sample Problem 11.3

Uniform Rectilinear-Motion

Uniformly Accelerated Rectilinear-Motion

Motion of Several Particles: Relative Motion

Sample Problem 11.4

Motion of Several Particles: Dependent Motion

Sample Problem 11.5

Graphical Solution of Rectilinear-Motion Problems

Other Graphical Methods

Curvilinear Motion: Position, Velocity & Acceleration

Derivatives of Vector Functions

Rectangular Components of Velocity and Acceleration

Motion Relative to a Frame in Translation

Tangential and Normal Components

Radial and Transverse Components

Sample Problem 11.10

Sample Problem 11.12
Introduction

- **Dynamics includes:**
  - *Kinematics*: study of the geometry of motion. Kinematics is used to relate displacement, velocity, acceleration, and time without reference to the cause of motion.
  - *Kinetics*: study of the relations existing between the forces acting on a body, the mass of the body, and the motion of the body. Kinetics is used to predict the motion caused by given forces or to determine the forces required to produce a given motion.

- **Rectilinear** motion: position, velocity, and acceleration of a particle as it moves along a straight line.

- **Curvilinear** motion: position, velocity, and acceleration of a particle as it moves along a curved line in two or three dimensions.
Rectilinear Motion: Position, Velocity & Acceleration

- Particle moving along a straight line is said to be in *rectilinear motion*.

- *Position coordinate* of a particle is defined by positive or negative distance of particle from a fixed origin on the line.

- The *motion* of a particle is known if the position coordinate for particle is known for every value of time $t$. Motion of the particle may be expressed in the form of a function, e.g.,

$$x = 6t^2 - t^3$$

or in the form of a graph $x$ vs. $t$. 

![Diagram of rectilinear motion](image)
Rectilinear Motion: Position, Velocity & Acceleration

• Instantaneous velocity may be positive or negative. Magnitude of velocity is referred to as particle speed.

• Consider particle which occupies position $P$ at time $t$ and $P'$ at $t + \Delta t$,

\[
\text{Average velocity} = \frac{\Delta x}{\Delta t}
\]

\[
\text{Instantaneous velocity} = v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}
\]

• Instantaneous velocity may be positive or negative. Magnitude of velocity is referred to as particle speed.

• From the definition of a derivative,

\[
v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}
\]

e.g., \[x = 6t^2 - t^3\]

\[
v = \frac{dx}{dt} = 12t - 3t^2
\]
Rectilinear Motion: Position, Velocity & Acceleration

- Consider particle with velocity $v$ at time $t$ and $v'$ at $t+\Delta t$,

$$\text{Instantaneous acceleration} = a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$$

- Instantaneous acceleration may be:
  - positive: increasing positive velocity or decreasing negative velocity
  - negative: decreasing positive velocity or increasing negative velocity.

- From the definition of a derivative,

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

e.g. $v = 12t - 3t^2$

$$a = \frac{dv}{dt} = 12 - 6t$$
• Consider particle with motion given by

\[ x = 6t^2 - t^3 \]

\[ v = \frac{dx}{dt} = 12t - 3t^2 \]

\[ a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = 12 - 6t \]

• at \( t = 0 \), \( x = 0, v = 0, \ a = 12 \ \text{m/s}^2 \)

• at \( t = 2 \ \text{s} \), \( x = 16 \ \text{m}, v = v_{\text{max}} = 12 \ \text{m/s}, \ a = 0 \)

• at \( t = 4 \ \text{s} \), \( x = x_{\text{max}} = 32 \ \text{m}, v = 0, \ a = -12 \ \text{m/s}^2 \)

• at \( t = 6 \ \text{s} \), \( x = 0, v = -36 \ \text{m/s}, \ a = 24 \ \text{m/s}^2 \)
Determination of the Motion of a Particle

• Recall, *motion* of a particle is known if position is known for all time $t$.

• Typically, conditions of motion are specified by the type of acceleration experienced by the particle. Determination of velocity and position requires two successive integrations.

• Three classes of motion may be defined for:
  - acceleration given as a function of *time*, $a = f(t)$
  - acceleration given as a function of *position*, $a = f(x)$
  - acceleration given as a function of *velocity*, $a = f(v)$
Determination of the Motion of a Particle

- Acceleration given as a function of time, \( a = f(t) \):
  \[
  \frac{dv}{dt} = a = f(t) \quad \Rightarrow \quad dv = f(t) \, dt \quad \Rightarrow \quad \int dv = \int f(t) \, dt \\
  v(t) - v_0 = t \int_0^t f(t) \, dt
  \\
  \frac{dx}{dt} = v(t) \quad \Rightarrow \quad dx = v(t) \, dt \quad \Rightarrow \quad \int dx = \int v(t) \, dt \\
  x(t) - x_0 = t \int_0^t v(t) \, dt
  \\
  \text{or} \quad \frac{dv}{dt} = a \quad \Rightarrow \quad \frac{dv}{v} = f(x) \, dx \\
  v \, dv = f(x) \, dx \quad \Rightarrow \quad \int_{v_0}^{v(x)} v \, dv = \int_{x_0}^{x} f(x) \, dx \\
  \frac{1}{2} v(x)^2 - \frac{1}{2} v_0^2 = \int_{x_0}^{x} f(x) \, dx
  \\
\]
Determination of the Motion of a Particle

- Acceleration given as a function of velocity, $a = f(v)$:

$$\frac{dv}{dt} = a = f(v) \quad \frac{dv}{f(v)} = dt \quad \int_{v_0}^{v(t)} \frac{dv}{f(v)} = \int_0^t dt$$

$$\int_{v_0}^{v(t)} \frac{dv}{f(v)} = t$$

$$v \frac{dv}{dx} = a = f(v) \quad dx = \frac{v dv}{f(v)} \quad \int_{x_0}^{x(t)} dx = \int_{v_0}^{v(t)} \frac{v dv}{f(v)}$$

$$x(t) - x_0 = \int_{v_0}^{v(t)} \frac{v dv}{f(v)}$$
Ball tossed with 10 m/s vertical velocity from window 20 m above ground.

Determine:
• velocity and elevation above ground at time \( t \),
• highest elevation reached by ball and corresponding time, and
• time when ball will hit the ground and corresponding velocity.

SOLUTION:
• Integrate twice to find \( v(t) \) and \( y(t) \).

• Solve for \( t \) at which velocity equals zero (time for maximum elevation) and evaluate corresponding altitude.

• Solve for \( t \) at which altitude equals zero (time for ground impact) and evaluate corresponding velocity.
Sample Problem 11.2

SOLUTION:
• Integrate twice to find \( v(t) \) and \( y(t) \).

\[
\frac{dv}{dt} = a = -9.81 \text{ m/s}^2
\]

\[
v(t) = \int dv = -\int_{0}^{t} 9.81 \, dt
\]

\[
v(t) - v_0 = -9.81t
\]

\[
v(t) = 10 \frac{\text{m}}{\text{s}} - \left(9.81 \frac{\text{m}}{\text{s}^2}\right)t
\]

\[
\frac{dy}{dt} = v = 10 - 9.81t
\]

\[
y(t) = \int_{0}^{t} dy = \int_{0}^{t} (10 - 9.81t) \, dt
\]

\[
y(t) - y_0 = 10t - \frac{1}{2} 9.81t^2
\]

\[
y(t) = 20 \text{ m} + \left(10 \frac{\text{m}}{\text{s}}\right)t - \left(4.905 \frac{\text{m}}{\text{s}^2}\right)t^2
\]
Sample Problem 11.2

- Solve for \( t \) at which velocity equals zero and evaluate corresponding altitude.

\[
v(t) = 10 \frac{m}{s} - \left(9.81 \frac{m}{s^2}\right)t = 0
\]

\[t = 1.019 \text{ s}\]

- Solve for \( t \) at which altitude equals zero and evaluate corresponding velocity.

\[
y(t) = 20 \text{ m} + \left(10 \frac{m}{s}\right)t - \left(4.905 \frac{m}{s^2}\right)t^2
\]

\[y = 20 \text{ m} + \left(10 \frac{m}{s}\right)(1.019 \text{ s}) - \left(4.905 \frac{m}{s^2}\right)(1.019 \text{ s})^2
\]

\[y = 25.1 \text{ m}\]
Sample Problem 11.2

• Solve for $t$ at which altitude equals zero and evaluate corresponding velocity.

$$y(t) = 20 \text{ m} + \left(10 \frac{\text{m}}{\text{s}}\right)t - \left(4.905 \frac{\text{m}}{\text{s}^2}\right)t^2 = 0$$

$t = -1.243 \text{ s}$ (meaningless $s$)
$t = 3.28 \text{ s}$

$$v(t) = 10 \frac{\text{m}}{\text{s}} - \left(9.81 \frac{\text{m}}{\text{s}^2}\right)t$$

$$v(3.28 \text{ s}) = 10 \frac{\text{m}}{\text{s}} - \left(9.81 \frac{\text{m}}{\text{s}^2}\right)(3.28 \text{ s})$$

$v = -22.2 \frac{\text{m}}{\text{s}}$
Brake mechanism used to reduce gun recoil consists of piston attached to barrel moving in fixed cylinder filled with oil. As barrel recoils with initial velocity $v_0$, piston moves and oil is forced through orifices in piston, causing piston and cylinder to decelerate at rate proportional to their velocity.

Determine $v(t)$, $x(t)$, and $v(x)$.

**SOLUTION:**

- Integrate $a = dv/dt = -kv$ to find $v(t)$.
- Integrate $v(t) = dx/dt$ to find $x(t)$.
- Integrate $a = v dv/dx = -kv$ to find $v(x)$. 

\[a = -kv\]
Sample Problem 11.3

SOLUTION:

- Integrate $a = dv/dt = -kv$ to find $v(t)$.

$$a = \frac{dv}{dt} = -kv$$

$$\int_{v_0}^{v(t)} \frac{dv}{v} = -k \int_{0}^{t} dt$$

$$\ln \frac{v(t)}{v_0} = -kt$$

$$v(t) = v_0 e^{-kt}$$

- Integrate $v(t) = dx/dt$ to find $x(t)$.

$$v(t) = \frac{dx}{dt} = v_0 e^{-kt}$$

$$\int_{0}^{t} dx = v_0 \int_{0}^{t} e^{-kt} dt$$

$$x(t) = v_0 \left[ -\frac{1}{k} e^{-kt} \right]_{0}^{t}$$

$$x(t) = \frac{v_0 k}{k} \left( 1 - e^{-kt} \right)$$
Sample Problem 11.3

- Integrate $a = v \frac{dv}{dx} = -kv$ to find $v(x)$.

$$a = v \frac{dv}{dx} = -kv \quad dv = -k \, dx$$

$$\int dv = -k \int_0^x dx$$

$$v - v_0 = -kx$$

- Alternatively,

$$v = v_0 - kx$$

with $$x(t) = \frac{v_0}{k} \left(1 - e^{-kt}\right)$$

and $$v(t) = v_0 e^{-kt} \quad \text{or} \quad e^{-kt} = \frac{v(t)}{v_0}$$

then $$x(t) = \frac{v_0}{k} \left(1 - \frac{v(t)}{v_0}\right)$$
For particle in uniform rectilinear motion, the acceleration is zero and the velocity is constant.

\[
\frac{dx}{dt} = v = \text{constant}
\]

\[
\int_{x_0}^{x} dx = v \int_{0}^{t} dt
\]

\[
x - x_0 = vt
\]

\[
x = x_0 + vt
\]
Uniformly Accelerated Rectilinear Motion

For particle in uniformly accelerated rectilinear motion, the acceleration of the particle is constant.

\[
\frac{dv}{dt} = a = \text{constant} \quad \int_{v_0}^{v} dv = a \int_{0}^{t} dt \quad v - v_0 = at \\
v = v_0 + at
\]

\[
\frac{dx}{dt} = v_0 + at \quad \int_{x_0}^{x} dx = \int_{0}^{t} (v_0 + at) dt \quad x - x_0 = v_0 t + \frac{1}{2} at^2 \\
x = x_0 + v_0 t + \frac{1}{2} at^2
\]

\[
v \frac{dv}{dx} = a = \text{constant} \quad \int_{v_0}^{v} v dv = a \int_{x_0}^{x} dx \quad \frac{1}{2} \left( v^2 - v_0^2 \right) = a(x - x_0) \\
v^2 = v_0^2 + 2a(x - x_0)
\]
Motion of Several Particles: Relative Motion

- For particles moving along the same line, time should be recorded from the same starting instant and displacements should be measured from the same origin in the same direction.

\[ x_{B/A} = x_B - x_A = \text{relative position of } B \text{ with respect to } A \]

\[ x_B = x_A + x_{B/A} \]

\[ v_{B/A} = v_B - v_A = \text{relative velocity of } B \text{ with respect to } A \]

\[ v_B = v_A + v_{B/A} \]

\[ a_{B/A} = a_B - a_A = \text{relative acceleration of } B \text{ with respect to } A \]

\[ a_B = a_A + a_{B/A} \]
Sample Problem 11.4

Ball thrown vertically from 12 m level in elevator shaft with initial velocity of 18 m/s. At same instant, open-platform elevator passes 5 m level moving upward at 2 m/s.

Determine (a) when and where ball hits elevator and (b) relative velocity of ball and elevator at contact.

SOLUTION:

• Substitute initial position and velocity and constant acceleration of ball into general equations for uniformly accelerated rectilinear motion.

• Substitute initial position and constant velocity of elevator into equation for uniform rectilinear motion.

• Write equation for relative position of ball with respect to elevator and solve for zero relative position, i.e., impact.

• Substitute impact time into equation for position of elevator and relative velocity of ball with respect to elevator.
Sample Problem 11.4

SOLUTION:

• Substitute initial position and velocity and constant acceleration of ball into general equations for uniformly accelerated rectilinear motion.

\[
v_B = v_0 + at = 18 \frac{\text{m}}{\text{s}} - \left(9.81 \frac{\text{m}}{\text{s}^2}\right)t
\]

\[
y_B = y_0 + v_0 t + \frac{1}{2} a t^2 = 12 \text{ m} + \left(18 \frac{\text{m}}{\text{s}}\right)t - \left(4.905 \frac{\text{m}}{\text{s}^2}\right)t^2
\]

• Substitute initial position and constant velocity of elevator into equation for uniform rectilinear motion.

\[
v_E = 2 \frac{\text{m}}{\text{s}}
\]

\[
y_E = y_0 + v_E t = 5 \text{ m} + \left(2 \frac{\text{m}}{\text{s}}\right)t
\]
Sample Problem 11.4

• Write equation for relative position of ball with respect to elevator and solve for zero relative position, i.e., impact.

\[ y_{B/E} = \left(12 + 18t - 4.905t^2\right) - (5 + 2t) = 0 \]

\[ t = -0.39 \text{ s (meaningless)} \]
\[ t = 3.65 \text{ s} \]

• Substitute impact time into equations for position of elevator and relative velocity of ball with respect to elevator.

\[ y_E = 5 + 2(3.65) \]
\[ y_E = 12.3 \text{ m} \]

\[ v_{B/E} = \left(18 - 9.81t\right) - 2 \]
\[ = 16 - 9.81(3.65) \]
\[ v_{B/E} = -19.81 \text{ m/s} \]
Motion of Several Particles: Dependent Motion

- Position of a particle may depend on position of one or more other particles.

- Position of block $B$ depends on position of block $A$. Since rope is of constant length, it follows that sum of lengths of segments must be constant.

  \[ x_A + 2x_B = \text{constant} \] (one degree of freedom)

- Positions of three blocks are dependent.

  \[ 2x_A + 2x_B + x_C = \text{constant} \] (two degrees of freedom)

- For linearly related positions, similar relations hold between velocities and accelerations.

  \[
  2 \frac{dx_A}{dt} + 2 \frac{dx_B}{dt} + \frac{dx_C}{dt} = 0 \quad \text{or} \quad 2v_A + 2v_B + v_C = 0
  \]

  \[
  2 \frac{dv_A}{dt} + 2 \frac{dv_B}{dt} + \frac{dv_C}{dt} = 0 \quad \text{or} \quad 2a_A + 2a_B + a_C = 0
  \]
Sample Problem 11.5

Pulley $D$ is attached to a collar which is pulled down at 3 in./s. At $t = 0$, collar $A$ starts moving down from $K$ with constant acceleration and zero initial velocity. Knowing that velocity of collar $A$ is 12 in./s as it passes $L$, determine the change in elevation, velocity, and acceleration of block $B$ when block $A$ is at $L$.

SOLUTION:

- Define origin at upper horizontal surface with positive displacement downward.
- Collar $A$ has uniformly accelerated rectilinear motion. Solve for acceleration and time $t$ to reach $L$.
- Pulley $D$ has uniform rectilinear motion. Calculate change of position at time $t$.
- Block $B$ motion is dependent on motions of collar $A$ and pulley $D$. Write motion relationship and solve for change of block $B$ position at time $t$.
- Differentiate motion relation twice to develop equations for velocity and acceleration of block $B$. 
Sample Problem 11.5

SOLUTION:

- Define origin at upper horizontal surface with positive displacement downward.

- Collar A has uniformly accelerated rectilinear motion. Solve for acceleration and time \( t \) to reach \( L \).

\[
v_A^2 = (v_A)_0^2 + 2a_A [x_A - (x_A)_0]
\]

\[
\left(12 \text{ in/s}\right)^2 = 2a_A (8 \text{ in.}) \quad a_A = 9 \text{ in/s}^2
\]

\[
v_A = (v_A)_0 + a_A t
\]

\[
12 \text{ in/s} = 9 \text{ in/s}^2 t \quad t = 1.333 \text{s}
\]
Sample Problem 11.5

• Pulley $D$ has uniform rectilinear motion. Calculate change of position at time $t$.

\[
x_D = (x_D)_0 + v_D t
\]

\[
x_D - (x_D)_0 = \left( 3 \text{ in.} \right) \left( 1.333 \text{ s} \right) = 4 \text{ in.}
\]

• Block $B$ motion is dependent on motions of collar $A$ and pulley $D$. Write motion relationship and solve for change of block $B$ position at time $t$.

Total length of cable remains constant,

\[
x_A + 2x_D + x_B = (x_A)_0 + 2(x_D)_0 + (x_B)_0
\]

\[
[x_A - (x_A)_0] + 2[x_D - (x_D)_0] + [x_B - (x_B)_0] = 0
\]

(8 in.) + 2(4 in.) + [x_B - (x_B)_0] = 0

\[
x_B - (x_B)_0 = -16 \text{ in.}
\]
Sample Problem 11.5

- Differentiate motion relation twice to develop equations for velocity and acceleration of block B.

\[ x_A + 2x_D + x_B = \text{constant} \]

\[ v_A + 2v_D + v_B = 0 \]

\[ \left( 12 \ \text{in.} \right) + 2 \left( 3 \ \text{in.} \right) + v_B = 0 \]

\[ a_A + 2a_D + a_B = 0 \]

\[ \left( 9 \ \text{in.}^2 \right) + v_B = 0 \]

\[ v_B = 18 \ \text{in.}/s \]

\[ a_B = -9 \ \text{in.}/s^2 \]
Graphical Solution of Rectilinear-Motion Problems

- Given the $x$-$t$ curve, the $v$-$t$ curve is equal to the $x$-$t$ curve slope.

- Given the $v$-$t$ curve, the $a$-$t$ curve is equal to the $v$-$t$ curve slope.
Given the $a-t$ curve, the change in velocity between $t_1$ and $t_2$ is equal to the area under the $a-t$ curve between $t_1$ and $t_2$.

Given the $v-t$ curve, the change in position between $t_1$ and $t_2$ is equal to the area under the $v-t$ curve between $t_1$ and $t_2$. 
Other Graphical Methods

- **Moment-area method** to determine particle position at time $t$ directly from the $a-t$ curve:

  $$x_1 - x_0 = \text{area under } v-t \text{ curve}$$

  $$= v_0 t_1 + \int_{v_0}^{v_1} (t_1 - t) dv$$

  using $dv = a \, dt$,

  $$x_1 - x_0 = v_0 t_1 + \int_{v_0}^{v_1} (t_1 - t) a \, dt$$

  $$\int_{v_0}^{v_1} (t_1 - t) a \, dt = \text{first moment of area under } a-t \text{ curve with respect to } t = t_1 \text{ line.}$$

  $$x_1 = x_0 + v_0 t_1 + (\text{area under } a-t \text{ curve})(t_1 - \bar{t})$$

  $$\bar{t} = \text{abscissa of centroid } C$$
Other Graphical Methods

- Method to determine particle acceleration from $v$-$x$ curve:

$$a = v \frac{dv}{dx}$$

$$= AB \tan \theta$$

$$= BC = \text{subnormal to } v$-$x$ curve$$
Curvilinear Motion: Position, Velocity & Acceleration

- Particle moving along a curve other than a straight line is in *curvilinear motion*.

- *Position vector* of a particle at time $t$ is defined by a vector between origin $O$ of a fixed reference frame and the position occupied by particle.

- Consider particle which occupies position $P$ defined by $r$ at time $t$ and $P'$ defined by $r'$ at $t + \Delta t$,

$$
\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}
$$

= instantaneous velocity (vector)

$$
v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}
$$

= instantaneous speed (scalar)
Curvilinear Motion: Position, Velocity & Acceleration

• Consider velocity $\vec{v}$ of particle at time $t$ and velocity $\vec{v}'$ at $t + \Delta t$,

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$$

= instantaneous acceleration (vector)

• In general, acceleration vector is not tangent to particle path and velocity vector.
Derivatives of Vector Functions

- Let \( \vec{P}(u) \) be a vector function of scalar variable \( u \),
  \[
  \frac{d\vec{P}}{du} = \lim_{\Delta u \to 0} \frac{\Delta \vec{P}}{\Delta u} = \lim_{\Delta u \to 0} \frac{\vec{P}(u + \Delta u) - \vec{P}(u)}{\Delta u}
  \]

- Derivative of vector sum,
  \[
  \frac{d(\vec{P} + \vec{Q})}{du} = \frac{d\vec{P}}{du} + \frac{d\vec{Q}}{du}
  \]

- Derivative of product of scalar and vector functions,
  \[
  \frac{d(f \vec{P})}{du} = \frac{df}{du} \vec{P} + f \frac{d\vec{P}}{du}
  \]

- Derivative of scalar product and vector product,
  \[
  \frac{d(\vec{P} \cdot \vec{Q})}{du} = \frac{d\vec{P}}{du} \cdot \vec{Q} + \vec{P} \cdot \frac{d\vec{Q}}{du}
  \]
  \[
  \frac{d(\vec{P} \times \vec{Q})}{du} = \frac{d\vec{P}}{du} \times \vec{Q} + \vec{P} \times \frac{d\vec{Q}}{du}
  \]
Rectangular Components of Velocity & Acceleration

- When position vector of particle $P$ is given by its rectangular components,
  \[ \vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \]

- Velocity vector,
  \[ \vec{v} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k} = \dot{x}\vec{i} + \dot{y}\vec{j} + \dot{z}\vec{k} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k} \]

- Acceleration vector,
  \[ \vec{a} = \frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\vec{j} + \frac{d^2z}{dt^2}\vec{k} = \ddot{x}\vec{i} + \ddot{y}\vec{j} + \ddot{z}\vec{k} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k} \]
Rectangular Components of Velocity & Acceleration

- Rectangular components particularly effective when component accelerations can be integrated independently, e.g., motion of a projectile,

\[ a_x = \ddot{x} = 0 \quad a_y = \ddot{y} = -g \quad a_z = \ddot{z} = 0 \]

with initial conditions,

\[ x_0 = y_0 = z_0 = 0 \quad (v_x)_0, (v_y)_0, (v_z)_0 = 0 \]

Integrating twice yields

\[ v_x = (v_x)_0 \quad v_y = (v_y)_0 - gt \quad v_z = 0 \]

\[ x = (v_x)_0 t \quad y = (v_y)_0 y - \frac{1}{2} gt^2 \quad z = 0 \]

- Motion in horizontal direction is uniform.
- Motion in vertical direction is uniformly accelerated.

- Motion of projectile could be replaced by two independent rectilinear motions.
Motion Relative to a Frame in Translation

- Designate one frame as the *fixed frame of reference*. All other frames not rigidly attached to the fixed reference frame are *moving frames of reference*.
- Position vectors for particles $A$ and $B$ with respect to the fixed frame of reference $Oxyz$ are $\vec{r}_A$ and $\vec{r}_B$.
- Vector $\vec{r}_{B/A}$ joining $A$ and $B$ defines the position of $B$ with respect to the moving frame $Ax'y'z'$ and
  \[ \vec{r}_B = \vec{r}_A + \vec{r}_{B/A} \]
- Differentiating twice,
  \[ \vec{v}_B = \vec{v}_A + \vec{v}_{B/A} \quad \vec{v}_{B/A} = \text{velocity of } B \text{ relative to } A. \]
  \[ \vec{a}_B = \vec{a}_A + \vec{a}_{B/A} \quad \vec{a}_{B/A} = \text{acceleration of } B \text{ relative to } A. \]
- Absolute motion of $B$ can be obtained by combining motion of $A$ with relative motion of $B$ with respect to moving reference frame attached to $A$. 
Tangential and Normal Components

- Velocity vector of particle is tangent to path of particle. In general, acceleration vector is not. Wish to express acceleration vector in terms of tangential and normal components.

- $\vec{e}_t$ and $\vec{e}_t'$ are tangential unit vectors for the particle path at $P$ and $P'$. When drawn with respect to the same origin, $\Delta \vec{e}_t = \vec{e}_t' - \vec{e}_t$ and $\Delta \theta$ is the angle between them.

$$\Delta e_t = 2 \sin(\Delta \theta/2)$$

$$\lim_{\Delta \theta \to 0} \frac{\Delta \vec{e}_t}{\Delta \theta} = \lim_{\Delta \theta \to 0} \frac{\sin(\Delta \theta/2)}{\Delta \theta/2} \vec{e}_n = \vec{e}_n$$

$$\vec{e}_n = \frac{d\vec{e}_t}{d\theta}$$
Tangential and Normal Components

- With the velocity vector expressed as \( \vec{v} = v \hat{e}_t \), the particle acceleration may be written as
  \[
  \vec{a} = \frac{d\vec{v}}{dt} = \frac{dv}{dt} \hat{e}_t + v \frac{d\hat{e}_t}{dt} + v \frac{d\hat{e}_n}{d\theta} \frac{d\theta}{ds} \frac{ds}{dt}
  \]
  but
  \[
  \frac{d\hat{e}_t}{d\theta} = \hat{e}_n \quad \rho \frac{d\theta}{ds} = ds \quad \frac{ds}{dt} = v
  \]

  After substituting,
  \[
  \vec{a} = \frac{dv}{dt} \hat{e}_t + \frac{v^2}{\rho} \hat{e}_n \quad a_t = \frac{dv}{dt} \quad a_n = \frac{v^2}{\rho}
  \]

- Tangential component of acceleration reflects change of speed and normal component reflects change of direction.
- Tangential component may be positive or negative. Normal component always points toward center of path curvature.
Tangential and Normal Components

• Relations for tangential and normal acceleration also apply for particle moving along space curve.

\[ \vec{a} = \frac{d\vec{v}}{dt} \vec{e}_t + \frac{v^2}{\rho} \vec{e}_n \quad a_t = \frac{d\vec{v}}{dt} \quad a_n = \frac{v^2}{\rho} \]

• Plane containing tangential and normal unit vectors is called the osculating plane.

• Normal to the osculating plane is found from

\[ \vec{e}_b = \vec{e}_t \times \vec{e}_n \]

\[ \vec{e}_n = \text{principal normal} \]

\[ \vec{e}_b = \text{binormal} \]

• Acceleration has no component along binormal.
Radial and Transverse Components

- When particle position is given in polar coordinates, it is convenient to express velocity and acceleration with components parallel and perpendicular to $OP$.

- The particle velocity vector is

$$\vec{v} = \frac{d}{dt} (r \vec{e}_r) = \frac{dr}{dt} \vec{e}_r + r \frac{d\vec{e}_r}{dt} = \frac{dr}{dt} \vec{e}_r + r \frac{d\theta}{dt} \vec{e}_\theta$$

$$= \dot{r} \vec{e}_r + r \dot{\theta} \vec{e}_\theta$$

- Similarly, the particle acceleration vector is

$$\vec{a} = \frac{d}{dt} \left( \frac{dr}{dt} \vec{e}_r + r \frac{d\theta}{dt} \vec{e}_\theta \right)$$

$$= \frac{d^2 r}{dt^2} \vec{e}_r + \frac{dr}{dt} \frac{d\vec{e}_r}{dt} + \frac{dr}{dt} \frac{d\theta}{dt} \vec{e}_\theta + r \frac{d^2 \theta}{dt^2} \vec{e}_\theta + r \frac{d\theta}{dt} \frac{d\vec{e}_\theta}{dt}$$

$$= \left(\ddot{r} - r \dot{\theta}^2\right) \vec{e}_r + \left(\ddot{\theta} + 2 \dot{r} \dot{\theta}\right) \vec{e}_\theta$$
Radial and Transverse Components

- When particle position is given in cylindrical coordinates, it is convenient to express the velocity and acceleration vectors using the unit vectors $\vec{e}_R$, $\vec{e}_\theta$, and $\vec{k}$.

- Position vector,
  \[
  \vec{r} = R \vec{e}_R + z \vec{k}
  \]

- Velocity vector,
  \[
  \vec{v} = \frac{d\vec{r}}{dt} = \dot{R} \vec{e}_R + R \dot{\theta} \vec{e}_\theta + z \vec{k}
  \]

- Acceleration vector,
  \[
  \vec{a} = \frac{d\vec{v}}{dt} = \left(\ddot{R} - R \dot{\theta}^2\right) \vec{e}_R + \left(R \ddot{\theta} + 2 \dot{R} \dot{\theta}\right) \vec{e}_\theta + \dddot{z} \vec{k}
  \]
A motorist is traveling on curved section of highway at 60 mph. The motorist applies brakes causing a constant deceleration rate.

Knowing that after 8 s the speed has been reduced to 45 mph, determine the acceleration of the automobile immediately after the brakes are applied.

**SOLUTION:**

- Calculate tangential and normal components of acceleration.

- Determine acceleration magnitude and direction with respect to tangent to curve.
Sample Problem 11.10

SOLUTION:

- Calculate tangential and normal components of acceleration.

\[
a_t = \frac{\Delta v}{\Delta t} = \frac{(66 - 88) \text{ ft/s}}{8 \text{ s}} = -2.75 \text{ ft/s}^2
\]

\[
a_n = \frac{v^2}{\rho} = \frac{(88 \text{ ft/s})^2}{2500 \text{ ft}} = 3.10 \text{ ft/s}^2
\]

- Determine acceleration magnitude and direction with respect to tangent to curve.

\[
a = \sqrt{a_t^2 + a_n^2} = \sqrt{(-2.75)^2 + 3.10^2} = 4.14 \text{ ft/s}^2
\]

\[
\alpha = \tan^{-1} \frac{a_n}{a_t} = \tan^{-1} \frac{3.10}{2.75} = 48.4^\circ
\]

60 mph = 88 ft/s
45 mph = 66 ft/s
Sample Problem 11.12

Rotation of the arm about O is defined by $\theta = 0.15t^2$ where $\theta$ is in radians and $t$ in seconds. Collar B slides along the arm such that $r = 0.9 - 0.12t^2$ where $r$ is in meters.

After the arm has rotated through $30^\circ$, determine (a) the total velocity of the collar, (b) the total acceleration of the collar, and (c) the relative acceleration of the collar with respect to the arm.

SOLUTION:

• Evaluate time $t$ for $\theta = 30^\circ$.

• Evaluate radial and angular positions, and first and second derivatives at time $t$.

• Calculate velocity and acceleration in cylindrical coordinates.

• Evaluate acceleration with respect to arm.
Sample Problem 11.12

**SOLUTION:**

- Evaluate time $t$ for $\theta = 30^\circ$.

  \[ \theta = 0.15 t^2 \]
  \[ = 30^\circ = 0.524 \text{ rad} \quad t = 1.869 \text{ s} \]

- Evaluate radial and angular positions, and first and second derivatives at time $t$.

  \[ r = 0.9 - 0.12 t^2 = 0.481 \text{ m} \]
  \[ \dot{r} = -0.24 t = -0.449 \text{ m/s} \]
  \[ \ddot{r} = -0.24 \text{ m/s}^2 \]
  \[ \theta = 0.15 t^2 = 0.524 \text{ rad} \]
  \[ \dot{\theta} = 0.30 t = 0.561 \text{ rad/s} \]
  \[ \ddot{\theta} = 0.30 \text{ rad/s}^2 \]
Sample Problem 11.12

- Calculate velocity and acceleration.

\[ v_r = \dot{r} = -0.449 \text{ m/s} \]
\[ v_\theta = r \dot{\theta} = (0.481 \text{ m})(0.561 \text{ rad/s}) = 0.270 \text{ m/s} \]
\[ v = \sqrt{v_r^2 + v_\theta^2} \]
\[ \beta = \tan^{-1} \frac{v_\theta}{v_r} \]
\[ v = 0.524 \text{ m/s} \]
\[ \beta = 31.0^\circ \]

\[ a_r = \ddot{r} - r \dot{\theta}^2 \]
\[ = -0.240 \text{ m/s}^2 - (0.481 \text{ m})(0.561 \text{ rad/s})^2 \]
\[ = -0.391 \text{ m/s}^2 \]

\[ a_\theta = r \ddot{\theta} + 2 \dot{r} \dot{\theta} \]
\[ = (0.481 \text{ m})(0.3 \text{ rad/s}^2) + 2(-0.449 \text{ m/s})(0.561 \text{ rad/s}) \]
\[ = -0.359 \text{ m/s}^2 \]

\[ a = \sqrt{a_r^2 + a_\theta^2} \]
\[ \gamma = \tan^{-1} \frac{a_\theta}{a_r} \]
\[ a = 0.531 \text{ m/s} \]
\[ \gamma = 42.6^\circ \]
Sample Problem 11.12

- Evaluate acceleration with respect to arm.

Motion of collar with respect to arm is rectilinear and defined by coordinate $r$.

$$a_{B/OA} = \ddot{r} = -0.240 \text{ m/s}^2$$