AXIOMS OF THE REAL NUMBER SYSTEM

What is a real number? To prove anything about real numbers you must firstly know what a real number is. For example, one can prove that there exists no rational number \( r = \frac{a}{b} \) \((a, b \in \mathbb{Z}, b \neq 0)\) such that \( r^2 = 2 \). But we can prove that there exists a unique positive real number \( a \) such that \( a^2 = 2 \) and then we can denote this unique positive real number \( a \) by \( \sqrt{2} \). What is missing in \( \mathbb{Q} \) that makes it impossible to define \( \sqrt{2} \) in \( \mathbb{Q} \)? And which property of \( \mathbb{R} \) enables us to prove that there exists a unique positive real number \( a \) such that \( a^2 = 2 \)? To prove this we must clearly know what the set \( \mathbb{R} \) of all real numbers is. How do we obtain your geometric idea of the real line as a complete continuous piece of points without holes? Indeed this real line \( \mathbb{R} \) contains ‘much much more’ points than these roots like \( \sqrt{2} \). The property of \( \mathbb{R} \) that enables us to prove the existence of \( \sqrt{2} \) is the ‘completeness’ of \( \mathbb{R} \) which we shall explain below.

Analysis depends on the real number system. So we must know clearly what real numbers are. The easier way is to start by assuming some axioms for the real number system and then construct everything starting from these axioms. This is what we shall do. But indeed, the real number system is constructed using the Axioms of Set Theory. The axiomatic set theory is the foundation of all mathematics. Among these axioms of set theory, there is an axiom which we can use to define the ‘infinite’ set \( \mathbb{N} = \{0, 1, 2, 3, \ldots\} \) of natural numbers. We also obtain the method of proof by induction: For a sequence \( P_0, P_1, P_2, \ldots, P_n, P_{n+1}, \ldots \) of statements, to prove that \( P_n \) is true for all \( n \in \mathbb{N} \), it suffices to prove that \( P_0 \) is true and to prove that for all \( k \in \mathbb{N} \), if \( P_k \) is true, then \( P_{k+1} \) is true. Using induction, we are then able to define addition and multiplication on \( \mathbb{N} \) and obtain its usual properties. Also we define the order relation \( < \) on \( \mathbb{N} \) and obtain its usual properties. The construction of the set \( \mathbb{N} \) of natural numbers and obtaining its properties which you are familiar from the primary school is the main step to start for the construction of number systems. The next easier step is to construct the set \( \mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} \) of integers with the operations addition and multiplication and with the order relation \( < \). After that by a general algebraic procedure one can construct the set \( \mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z} \text{ and } n \neq 0\} \) of rational numbers with the operations addition and multiplication and with the order relation \( < \). Then comes the main idea of analysis: “completion” of \( \mathbb{Q} \) to obtain the set \( \mathbb{R} \) of real numbers with the operations addition and multiplication and with the order relation \( < \). “Completeness” of \( \mathbb{R} \) is the property that makes it different from \( \mathbb{Q} \); that is the part which we need when considering the geometric real line without any holes. This is the property that enables us to work with limits, to define derivatives, integrals etc. The real line has no missing “limit” holes. You will learn the precise meaning of all those in your courses on analysis. You will also learn how we make a “completion” of a “metric space”. Every mathematician should once learn the details of this construction of number systems starting from the axioms of set theory. It is not easy and it contains some important ideas and details to overcome. But as we said above we shall choose the easier way of giving a set of axioms for the real number system.

We assume that there exists a set \( \mathbb{R} \) (which we shall call the set of all real numbers) with two operations addition(+) and multiplication(\cdot) on \( \mathbb{R} \) and with an order relation \( < \) that make \( \mathbb{R} \) a complete ordered field. What does this mean? We shall give the meaning of \( \mathbb{R} \) being a complete ordered field in three groups of axioms. We shall give a minimal set of axioms that suffices to obtain all the usual properties of \( \mathbb{R} \). Moreover, these set of axioms characterize \( \mathbb{R} \) uniquely; that is, we can prove that any complete ordered field is necessarily isomorphic to \( \mathbb{R} \) (isomorphism as an ordered field, that is, the isomorphism function preserves the order as well as the operations addition and multiplication). To learn what all these means precisely see for example the book Calculus by Michael Spivak (Corrected 3rd ed., Cambridge University Press, 2006).

A) Field properties of \( \mathbb{R} \): \( \mathbb{R} \) is a field with the operations addition(+) and multiplication(\cdot). These axioms will give us all the usual algebraic properties of \( \mathbb{R} \) that you are already familiar.

B) Ordered field properties of \( \mathbb{R} \): These will give us the usual properties of the order relation \( < \) that you are familiar with. Remember the properties of the inequalities related with adding a real number to both sides of an inequality and multiplying both sides of an inequality by a real number.

C) Completeness of \( \mathbb{R} \) (the Least Upper Bound Property of \( \mathbb{R} \)): This is the property that makes \( \mathbb{R} \) different from \( \mathbb{Q} \). The properties in part (A) and (B) are also possessed by \( \mathbb{Q} \) but \( \mathbb{Q} \) does not have completeness. We shall explain below its meaning.
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There exists a set \( \mathbb{R} \) (which we shall call the set of all real numbers) with two operations addition(+) and multiplication(\cdot) on \( \mathbb{R} \) and with a relation < on \( \mathbb{R} \) that satisfy the below axioms:

A) FIELD AXIOMS:

(i) **Associativity:** For all \( a, b, c \in \mathbb{R} \), \( a + (b + c) = (a + b) + c \) and \( a(bc) = (ab)c \).

(ii) **Commutativity:** For all \( a, b \in \mathbb{R} \), \( a + b = b + a \) and \( ab = ba \).

(iii) **The zero element 0 (additive identity) and additive inverses:** There exists an element in \( \mathbb{R} \), denoted by 0, such that:
   - For all \( a \in \mathbb{R} \), \( a + 0 = a \).
   - For each \( a \in \mathbb{R} \), there exists a unique element in \( \mathbb{R} \), denoted by \(-a\), such that \( a + (-a) = 0 \).

(iv) **The multiplicative identity 1 and multiplicative inverses of nonzero elements:** There exists a nonzero element in \( \mathbb{R} \), denoted by 1, such that:
   - For all \( a \in \mathbb{R} \), \( a \cdot 1 = a \).
   - For each \( a \in \mathbb{R} \), if \( a \neq 0 \), then there exists a unique element in \( \mathbb{R} \), denoted by \( a^{-1} \), such that \( a \cdot a^{-1} = 1 \).

(v) **Distributivity of multiplication over addition:** For all \( a, b, c \in \mathbb{R} \), \( a(b + c) = ab + ac \).

B) ORDER AXIOMS:

(i) **Transitivity:** For all \( a, b, c \in \mathbb{R} \), if \( a < b \) and \( b < c \), then \( a < c \).

(ii) **Trichotomy Law:** For every \( a, b \in \mathbb{R} \), one and only one of the following is true: \( a < b \) or \( a = b \) or \( b < a \).

(iii) **Relation of order with the operations addition and multiplication:**
   For every \( a, b, c \in \mathbb{R} \), we have:
   - If \( a < b \), then \( a + c < b + c \).
   - If \( a < b \) and \( c > 0 \), then \( ac < bc \).
   - If \( a < b \) and \( c < 0 \), then \( bc < ac \).

C) THE LEAST UPPER BOUND AXIOM (COMPLETEENESS AXIOM):

For every nonempty subset \( A \) of \( \mathbb{R} \), if \( A \) is bounded from above, then \( A \) has a least upper bound, that is, the set \( U = \{ u \in \mathbb{R} | u \text{ is an upper bound of } A \} \) of all upper bounds of \( A \) has a least (smallest) element which means that there exists an upper bound \( u_0 \) of \( A \) such that \( u_0 \leq u \) for every upper bound \( u \) of \( A \).

Note that for every \( a, b \in \mathbb{R} \), we define \( a \leq b \) if \( a < b \) or \( a = b \); \( a > b \) if \( b < a \); and \( a \geq b \) if \( b < a \) or \( a = b \).

Let us briefly explain the Least Upper Bound Axiom. Let \( A \) be a nonempty subset of \( \mathbb{R} \). Suppose that \( A \) is bounded from above. This means \( A \) has an upper bound in \( \mathbb{R} \), that is, there exists an element \( v \) in \( \mathbb{R} \) such that for all \( a \in A \), we have \( a \leq v \). Now consider the set of all upper bounds of \( A \), that is the set

\[
U = \{ u \in \mathbb{R} | u \text{ is an upper bound of } A \} = \{ u \in \mathbb{R} | u \geq a \text{ for all } a \in A \}.
\]

Since \( v \in U \), the set \( U \) is nonempty. The least upper bound axiom guarantees that this set \( U \) has a least (smallest, minimum) element. That is, there exists \( u_0 \in U \) such that \( u_0 \leq u \) for all \( u \in U \). This means, there exists a real number \( u_0 \) such that

- \( u_0 \) is an upper bound of \( A \), and
- \( u_0 \leq u \) for every upper bound \( u \) of \( A \).

Such an element \( u_0 \) is necessarily unique (why?). This necessarily unique element \( u_0 \) is denoted by \( \text{sup}(A) \) (read "supremum of \( A \)"):

\[
\text{sup}(A) = \text{the least upper bound of the nonempty set } A \text{ that is bounded from above}.
\]

It is the elegance of mathematics that all theorems of analysis is just proved by using the above axioms of the real number system: a huge structure of theorems is built just by the above axioms on one page. There is nothing else needed to prove the results of analysis (except of course the laws of logic and the axioms of set theory). These axioms form the basis of what we assume for real numbers; nothing else is needed.