1 Parametrization of Curves

Parametric Equations

If \( x = f(t) \) and \( y = g(t) \) are functions of the one parameter \( t \) when \( t \) runs over an interval \( I \), then the set of points

\[
C = \{(x, y) \mid (x, y) = (f(t), g(t)) \text{ for some } t \in I\}
\]

is called a parametric curve in the \( xy \)-plane and the curve \( C \) is said to be parametrized by the parametric equations \( x = f(t) \) and \( y = g(t) \) (they are called a parametrization of the curve \( C \)).

See your textbook for examples of parametric curves (Sections 10.1 and 10.2 of Chapter 10 Parametric Equations and Polar Coordinates).

Tangent Lines to a Parametric Curve

Suppose that for the parametrized curve

\[
x = f(t) \quad \text{and} \quad y = g(t),
\]

The tangent line to the parametric curve at \( t = t_0 \) is given by

\[
y = y(t_0) + \left( \frac{dy}{dt} \right)_{t=t_0}(x - x(t_0)),
\]

where \( \frac{dy}{dt} \) is the derivative of the function \( y = g(t) \) with respect to \( t \).
the functions $f$ and $g$ are differentiable on an open interval $I$ with
\[ \frac{dx}{dt} = f'(t) \neq 0 \quad \text{for all } t \in I. \]

Let $t_0 \in I$. Consider the point $P_0 = (x_0, y_0) = (f(t_0), g(t_0))$ on the parametrized curve. We want to find the tangent line to this curve at the point $P_0 = (x_0, y_0)$. For this we must express $y$ as a function of $x$ and then find the derivative $\frac{dy}{dx}$ at the point $x_0$. How to do this if it is not easy to find a formula for $y$ in terms of $x$ by eliminating $t$? Can this be done always? Can we find the answer just by working with $x = f(t)$ and $y = g(t)$ as functions of $t$ and finding some derivatives with respect to $t$ and evaluating at $t_0$? The answer is yes as explained below.

Finding $\frac{dy}{dx}$ if $\frac{dx}{dt} \neq 0$:
\[ \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}. \]

By the Intermediate Value Theorem for Derivative, our assumption that $\frac{dx}{dt} = f'(t) \neq 0$ for all $t$ in the open interval $I$ implies that either $\frac{dx}{dt} = f'(t) > 0$ for all $t \in I$ or $\frac{dx}{dt} = f'(t) < 0$ for all $t \in I$. So $x = f(t)$ is either an increasing function or a decreasing function on the open interval $I$. In any case, $x = f(t)$ is a one-to-one function on $I$ and hence it has an inverse function. That is, in theory, we can solve for $t$ in terms of $x$ (even if this "solution" is not expressible by our basic library of functions):
\[ x = f(t) \quad \text{and} \quad t = f^{-1}(x). \]

Now substituting $t = f^{-1}(x)$ into $y = g(t)$, we "eliminate" $t$ and express $y$ in terms of $x$, that is, $y = g(t) = g(f^{-1}(x)) = (g \circ f^{-1})(x)$ can be considered as a function of $x$, and then by the Chain Rule and the Rule for the Derivative of Inverse Functions,
\[ \frac{dy}{dx} = (g \circ f^{-1})'(x) = g'(f^{-1}(x)) \cdot (f^{-1})'(x) \]
\[ = g'(f^{-1}(x)) \cdot \frac{1}{f'(f^{-1}(x))} = \frac{g'(f^{-1}(x))}{f'(f^{-1}(x))} = \frac{g'(t)}{f'(f^{-1}(x))} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dx}. \]

since $t = f^{-1}(x)$. Or in terms of Leibnitz notation for derivative
\[ \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{1}{\frac{dx}{dt}} \cdot \frac{dy}{dt} = \frac{dy}{dx}. \]

As a result
\[ \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \quad \text{if } \frac{dx}{dt} \neq 0 \text{ for all } t \in I. \]

Tangent line at the point $P_0 = (x_0, y_0) = (f(t_0), g(t_0))$

Then the tangent line to the parametric curve $(x, y) = (f(t), g(t))$ at the point $P_0 = (x_0, y_0) = (f(t_0), g(t_0))$ is the line passing through the point $P_0 = (x_0, y_0)$ and having slope
\[ m_0 = \left. \frac{dy}{dx} \right|_{x_0} = \left. \frac{dy}{dt} \cdot \frac{dt}{dx} \right|_{t_0}. \]

The equation of the tangent line at $P_0 = (x_0, y_0)$ is then
\[ y - y_0 = m_0(x - x_0). \]

See your textbook for examples for finding tangent lines to parametric curves.
Finding $\frac{d^2y}{dx^2}$ if $\frac{dx}{dt} \neq 0$: $\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$ where $y' = \frac{dy}{dx}$.

Further if we assume that $f$ and $g$ have also second order derivatives on the open interval $I$, then we can also express $\frac{d^2y}{dx^2}$ in terms of $t$ similarly.

Let $y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$.

Then by the above results, we obtain

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy/dx}{dx/dt} \right) = \frac{d}{dx} \left( \frac{dy/dt}{dx/dt} \right) = \frac{d}{dt} \left( \frac{dy/dt}{dx/dt} \right)$$

\section{Graphing Parametrized Curves}

How to graph parametrized curves?

See Section 10.1 and 10.2 of your textbook for examples on graphing parametric curves. Given a parametric curve $x = f(t)$, $y = g(t)$, using your knowledge of graphing functions of one variable, you shall determine how $x = f(t)$ changes, on which intervals it is increasing and on which intervals it is decreasing. Similarly you shall determine how $y = g(t)$ changes. Combining these information on $x = f(t)$ and $y = g(t)$, you shall be able to graph the parametric curve. Of course, you shall also have some idea about how the graph is by giving some values to $t$ and finding the points $(x, y) = (f(t), g(t))$. This is the general idea for graphing parametric curves which is surely not an easy task.

Sometimes you can easily eliminate $t$ and find an equation for $x$ and $y$ which is a familiar curve that you know (like for example ellipse, hyperbola, parabola). See Section 10.1 in your textbook for some examples.

But in many cases, it is better to work with the parametrization $t$. For example, for the cycloid curve

$$x = a(t - \sin t), \quad y = a(1 - \cos t), \quad 0 \leq t \leq 2\pi,$$

where $a > 0$ is a constant, you can not solve for $t$ in terms of $x$ using our basic library of functions, and so you may not express $y$ as a function of $x$, although this exists in theory. But we may ask you to find the area of the region between the cycloid curve and $x$-axis, or we may ask you the slope of the tangent line at a point on the cycloid curve, and you must be able to find this using the parametrization in terms of $t$ without knowing the function expressing $y$ in terms of $x$.

\section{Length of Parametrized Curves and Arc Length Differential}

$$ds = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}$$

Length of Smooth Curves in the Plane

A parametric curve $C$

$$x = f(t), \quad y = g(t), \quad a \leq t \leq b,$$
is said to be a smooth curve if \( f \) and \( g \) are continuously differentiable on the interval \([a, b]\) and \((f'(t), g'(t)) \neq (0, 0)\) for all \( t \in [a, b] \) (that is for each \( t \in \mathbb{R} \), either \( f'(t) \neq 0 \) or \( g'(t) \neq 0 \), and so \([f'(t)]^2 + [g'(t)]^2 > 0\) for all \( t \in [a, b] \)).

If such a smooth curve \( C \) is traversed exactly once as \( t \) increases from \( t = a \) to \( t = b \), then the length of \( C \) is given by

\[
L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2} \, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt.
\]

To see why this seems reasonable for the length of the curve see Section 10.2 your textbook. Indeed, this is a theorem that is proved if one gives the definition of the length of parametrized curves which is a topic that you shall see in your analysis course.

**Arc Length Parameter of Smooth Curves in the Plane**

For the above smooth curve \( C \), the arc length parameter \( s = s(t) \) is defined by

\[
s = s(t) = \int_a^t \sqrt{[f'(z)]^2 + [g'(z)]^2} \, dz, \quad a \leq t \leq b.
\]

By the Fundamental Theorem of Calculus, \( \frac{ds}{dt} = \sqrt{[f'(t)]^2 + [g'(t)]^2} \) for all \( t \in [a, b] \). Since \( C \) is a smooth curve, \([f'(t)]^2 + [g'(t)]^2 > 0\) for all \( t \in [a, b] \). So \( \frac{ds}{dt} > 0 \) for all \( t \in [a, b] \) which implies that \( s = s(t) \) is an increasing function of \( t \), so it is a one-to-one function and has an inverse function. Thus, in theory, one can solve for \( t \) in terms of \( s \).

The arc length differential \( ds \) is then

\[
ds = \frac{ds}{dt} \, dt = \sqrt{[f'(t)]^2 + [g'(t)]^2} \, dt = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt
\]

Then the length of the curve \( C \) is

\[
L = \int_{t=a}^{t=b} ds = \int_a^b \frac{ds}{dt} \, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt.
\]

**Area of Surface of Revolution for Parametrized Curves**

If the smooth curve \( C \)

\[
x = f(t), \quad y = g(t), \quad a \leq t \leq b,
\]

is traversed exactly once as \( t \) increases from \( a \) to \( b \), then the areas of the surfaces generated by revolving the curve about the coordinate axes are as follows:

If the curve is revolved about the \( x \)-axis and \( y = g(t) \geq 0 \) for all \( t \in [a, b] \), then the surface area is

\[
S = \int_{t=a}^{t=b} 2\pi y \, ds = \int_{t=a}^{t=b} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt
\]

\[
= \int_a^b 2\pi g(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt
\]
If the curve is revolved about the y-axis and \( x = f(t) \geq 0 \) for all \( t \in [a, b] \), then the surface area is

\[
S = \int_{t=a}^{t=b} 2\pi x \, ds = \int_{t=a}^{t=b} 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt
\]

\[
= \int_{a}^{b} 2\pi f(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt.
\]

4 Polar Curves, Graphing in Polar Coordinates

**Polar curve** \( r = f(\theta) \): the parametrized curve \( x = r \cos \theta = f(\theta) \cos \theta, \) \( y = r \sin \theta = f(\theta) \sin \theta \)

For polar coordinates, polar equations and examples of polar curves, how to check symmetry for polar curves, see the lecture notes and Sections 10.3 and 10.4 of your textbook. You must have no difficulty in recognizing the polar equations of circles and lines (lines through origin, vertical lines and all other lines).

A polar curve \( r = f(\theta) \) means just the parametrized curve

\[
x = x = r \cos \theta = f(\theta) \cos \theta, \quad y = r \sin \theta = f(\theta) \sin \theta.
\]

As usual in our calculus course, either the domain of \( f(\theta) \) is given, or if it is not given, then it is understood that the domain of \( f \) consists of all \( \theta \in \mathbb{R} \) where the given formula for \( f(\theta) \) is defined.

So we apply our technique of parametrized curves to polar curves.

**Slope of a Polar Curve** \( r = f(\theta) \)

For the polar curve \( r = f(\theta) \), if \( f \) is a differentiable function on an open interval \( I \) and \( \frac{dx}{d\theta} \neq 0 \) for all \( \theta \in I \), then since we have the parametrized curve \( x = r \cos \theta = f(\theta) \cos \theta \) and \( y = r \sin \theta = f(\theta) \sin \theta \), we obtain that the slope of the polar curve is

\[
\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta) \sin \theta + f(\theta) \cos \theta}{f'(\theta) \cos \theta - f(\theta) \sin \theta}.
\]

If \( f(\theta_0) = 0 \) for some \( \theta_0 \in I \), then the polar curve \( r = f(\theta) \) passes through the origin at \( \theta = \theta_0 \) and the slope at \( \theta = \theta_0 \) is

\[
\left. \frac{dy}{dx} \right|_{\theta_0} = \frac{f'(\theta_0) \sin \theta_0 + f(\theta_0) \cos \theta_0}{f'(\theta_0) \cos \theta_0 - f(\theta_0) \sin \theta_0} = \frac{f'(\theta_0) \sin \theta_0}{f'(\theta_0) \cos \theta_0 - f(\theta_0) \sin \theta_0} = \tan \theta_0.
\]

**Length of a Polar Curve** \( r = f(\theta), \) \( \alpha \leq \theta \leq \beta \)

Let \( C \) be the polar curve \( r = f(\theta), \) \( \alpha \leq \theta \leq \beta \), where \( f \) is continuously differentiable on \( I \) and the curve \( C \) is traversed exactly once as \( \theta \) runs from \( \alpha \) to \( \beta \). The polar curve is just the parametrized curve

\[
x = r \cos \theta = f(\theta) \cos \theta, \quad y = r \sin \theta = f(\theta) \sin \theta, \quad \alpha \leq \theta \leq \beta.
\]

Show that

\[
\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2.
\]

So the length of the polar curve $C$ is

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \, d\theta = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta.$$ 

**Area of Fan-Shaped Region bounded by** $r = f(\theta) \geq 0$ **and** $\theta = \alpha, \theta = \beta$

The area of the fan-shaped region bounded by the polar curve $r = f(\theta)$, $\alpha \leq \theta \leq \beta$, and the lines $\theta = \alpha$ and $\theta = \beta$ is given by

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta = \int_{\alpha}^{\beta} \frac{1}{2} [f(\theta)]^2 \, d\theta,$$

when $f$ is a continuous function on $[\alpha, \beta]$.

See the lecture notes and Section 10.5 of your textbook for examples.

## 5 Exercises for Parametric Curves and Polar Curves

See the lecture notes and Chapter 10 of your textbook for examples and exercises on parametric curves and polar curves. For exercises, see Homework 10. Take also a copy of the notes given by Celal Cem Sarıoğlu on polar coordinates; it contains many examples with solutions.

## References


