Contents

1 Intermediate Value Theorem for Derivative 1
2 Cauchy’s Mean Value Theorem and L’Hôpital’s Rule 2
3 Finding absolute maximum and absolute minimum on $[a,b]$ 4
4 Second Derivative Test for Local Extrema 5
5 Application Problems 6
6 Newton’s Method for finding roots of $f(x) = 0$ 7

1 Intermediate Value Theorem for Derivative

Intermediate Value Theorem for Continuous Functions

Remember the Intermediate Value Theorem for Continuous Functions:

**Theorem 1.** If $f$ is a continuous function on the bounded closed interval $[a,b]$, where $a < b$ in $\mathbb{R}$, and if $y_0$ is a real number strictly between $f(a)$ and $f(b)$, then there exists $x_0$ strictly between $a$ and $b$ such that $f(x_0) = y_0$.

For example, we have been using this to find the existence of a root of an equation $f(x) = 0$ when $f$ is a continuous function on $[a,b]$ such that $f(a) < y_0 = 0 < f(b)$ or $f(a) > y_0 = 0 > f(b)$. In this case, the intermediate value theorem gives the existence of a point $x_0$ strictly between $a$ and $b$ such that $f(x_0) = 0$.

Intermediate Value Theorem for Derivative

**Theorem 2.** Let $f$ be a differentiable function on an open interval $I$. If $y_0$ is a real number that is strictly between $f'(a)$ and $f'(b)$ for some $a < b$ in the open interval $I$, then there exists $x_0$ strictly between $a$ and $b$ such that $f'(x_0) = y_0$. 


Remark.
If we have assumed that the derivative $f'$ is a continuous function on the interval $I$, then this result would follow from the **Intermediate Value Theorem for Continuous Functions**. But in the above theorem, continuity of $f'$ is not assumed. You shall see a proof of this theorem in your Analysis course next year.

Problem.
Find a function $f$ that is differentiable on an open interval $I$ such that $f'$ is not continuous at a point in $I$. This will show you that the above theorem is really meaningful since there are differentiable functions $f$ on an open interval $I$ (that is $f'(x)$ exists for all $x \in I$) but $f'$ is not continuous at some points in $I$.

**Corollary 3.** If $f$ is a differentiable function on an open interval $I$, and if $f'(x) \neq 0$ for all $x \in I$, then either $f$ is increasing on $I$ or $f$ is decreasing on $I$.

This is because if there exists points $a < b$ in $I$ such that $f'(a) < y_0 = 0 < f'(b)$ or $f'(a) > y_0 = 0 > f'(b)$, then by the Intermediate Value Theorem for Derivative, there exists $x_0$ strictly between $a$ and $b$ such that $f'(x_0) = 0$ which contradicts our hypothesis $f'(x) \neq 0$ for all $x \in I$. So either $f'(x) > 0$ for all $x \in I$ or $f'(x) < 0$ for all $x \in I$. Hence either $f$ is increasing on $I$ or $f$ is decreasing on $I$.

2  Cauchy’s Mean Value Theorem and L’Hôpital’s Rule

**Mean Value Theorem for Derivative**
Remember the important theorem for derivative that we have proved using Rolle’s Theorem in the previous week:

**(Lagrange’s) Mean Value Theorem:**
If $f$ is a continuous function on the bounded closed interval $[a, b]$ that is differentiable on the open interval $(a, b)$, where $a < b$ in $\mathbb{R}$, then there exists a point $c$ in the open interval $(a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

**Cauchy’s Mean Value Theorem**

**Theorem 4.** If $f$ and $g$ are continuous functions on the bounded closed interval $[a, b]$ that are differentiable on the open interval $(a, b)$, where $a < b$ in $\mathbb{R}$, then there exists a point $c$ in the open interval $(a, b)$ such that

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c),$$

and if moreover we assume that $g'(x) \neq 0$ for all $x \in (a, b)$, then necessarily $g(b) \neq g(a)$ and

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

**Proof.** Apply Rolle’s Theorem to the function

$$h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x), \quad a \leq x \leq b.$$
Then $h$ is a continuous function on the bounded closed interval $[a, b]$ that is differentiable on the open interval $(a, b)$ with $h'(x) = [f(b) - f(a)]g'(x) - [g(b) - g(a)]f'(x)$ for all $x \in (a, b)$. We also have

$$h(a) = f(b)g(a) - g(b)f(a) = h(b).$$

So by Rolle’s Theorem (or Mean Value Theorem), there exists a point $c$ in the open interval $(a, b)$ such that

$$h'(c) = 0,$$

which gives the result of the theorem. 

\[ \square \]

**L'Hôpital's Rule for the indeterminate form $\frac{0}{0}$**

**Theorem 5.** Let $a \in \mathbb{R}$ and suppose that $f$ and $g$ are differentiable functions on the open interval $(a - \delta_0, a + \delta_0) \setminus \{a\}$ for some $\delta_0 > 0$. That is, $f'(x)$ and $g'(x)$ exists for all $x \in (a - \delta_0, a + \delta_0)$ maybe except for $x = a$. Suppose that the following three conditions hold:

1. $g'(x) \neq 0$ for all $x \in (a - \delta_0, a + \delta_0) \setminus \{a\}$.
2. $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$.
3. $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists.

Then $g(x) \neq 0$ for all $x \in (a - \delta_0, a + \delta_0) \setminus \{a\}$ and $\lim_{x \to a} \frac{f(x)}{g(x)}$ exists with

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

See your textbook for a proof of this (at the end of Section 4.5 Indeterminate Forms and L’Hôpital’s Rule at pages 251–252). The following version of L’Hôpital’s Rule for $\frac{\pm \infty}{\pm \infty}$ is a bit harder (which you shall learn in your Analysis course next year).

**L'Hôpital's Rule for the indeterminate form $\frac{\pm \infty}{\pm \infty}$**

**Theorem 6.** Let $a \in \mathbb{R}$ and suppose that $f$ and $g$ are differentiable functions on the open interval $(a - \delta_0, a + \delta_0) \setminus \{a\}$ for some $\delta_0 > 0$. That is, $f'(x)$ and $g'(x)$ exists for all $x \in (a - \delta_0, a + \delta_0)$ maybe except for $x = a$. Suppose that the following three conditions hold:

1. $g'(x) \neq 0$ for all $x \in (a - \delta_0, a + \delta_0) \setminus \{a\}$.
2. $\lim_{x \to a} f(x) = \pm \infty = \lim_{x \to a} g(x)$.
3. $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists.

Then $g(x) \neq 0$ for all $x \in (a - \delta_0, a + \delta_0) \setminus \{a\}$ and $\lim_{x \to a} \frac{f(x)}{g(x)}$ exists with

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$
L'Hôpital's Rule

In the above theorems for L'Hôpital’s Rule, instead of $x \to a$ for some $a \in \mathbb{R}$, one can also take the following (with suitable modifications in the theorem for the domains of the functions):

$x \to a$, $x \to a^+$, $x \to a^-$, $x \to \infty$, $x \to -\infty$

Note also that in the above theorems the result is true also when $\lim_{x \to a} f'(x) = \pm \infty$, that is, these theorems hold if $\lim_{x \to a} f'(x) = L$ for some real number $L$ or $\lim_{x \to a} f'(x) = \pm \infty$.

In all cases, the above theorems clearly state the necessary hypothesis needed for writing $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$.

L'Hôpital's Rule is only for $0/0$ and $\pm \infty/\pm \infty$

For finding limits $\lim_{x \to a} f(x)/g(x)$, L'Hôpital’s Rule is used only if you have the indeterminate forms $0/0$ and $\pm \infty/\pm \infty$, that is, only if

$$\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x) \quad \text{or} \quad \lim_{x \to a} f(x) = \pm \infty = \lim_{x \to a} g(x).$$

For limits in the indeterminate forms

$$\pm \infty \cdot 0, \quad \infty - \infty, \quad 1^\infty, \quad \infty^0, \quad 0^0$$

you must firstly make some change to put the limit in a form that becomes $0/0$ or $\pm \infty/\pm \infty$ indeterminate form to which you can then apply L'Hôpital’s Rule. See the lecture notes and your textbook for examples of limits in these indeterminate forms.

3 Finding absolute maximum and absolute minimum on $[a, b]$

Extreme Value Theorem for continuous functions on $[a, b]$

**Theorem 7.** If $f$ is a function that is continuous on a bounded closed interval $[a, b]$, where $a < b$ in $\mathbb{R}$, then $f$ attains its absolute maximum value and absolute minimum value on $[a, b]$, that is, there exist $x_1, x_2 \in [a, b]$ such that

$$f(x_1) \leq f(x) \leq f(x_2) \quad \text{for all } x \in [a, b].$$

Thus $f(x_1)$ is the absolute minimum value of $f$ on $[a, b]$ and $f(x_2)$ is the absolute maximum value of $f$ on $[a, b]$.

The Extreme Value Theorem gives the existence of the absolute maximum and absolute minimum values of a continuous function on a bounded closed interval $[a, b]$. Existence theorems are very important in mathematics.

Note that the Extreme Value Theorem is not true in general if we take an open interval $(a, b)$ or if we take a function $f$ that is not continuous. In these cases, the existence is not guaranteed but still we can use our differentiation technique on the intervals where it is
continuous to determine where it is increasing or decreasing, and where it has local maximum and minimum values.

If $f$ is a function that is continuous on a bounded closed interval $[a, b]$, then using derivative, we shall find these existing values, that is the absolute maximum and absolute minimum values of $f$ on the bounded closed interval $[a, b]$ as follows.

Finding absolute maximum and absolute minimum of a continuous function on $[a, b]$

Let $f$ be a function that is continuous on a bounded closed interval $[a, b]$, where $a < b$ in $\mathbb{R}$. The Extreme Value Theorem gives the existence of the absolute maximum and absolute minimum values of $f$ on $[a, b]$.

The absolute extremum values of $f$ may be at the endpoints of the interval $[a, b]$ or at an interior point of $[a, b]$, that is, in the open interval $(a, b)$. If we have absolute extremum value at an interior point of $[a, b]$, that is at a point $c$ in the open interval $(a, b)$, then this point will also be clearly a local extremum value. So the absolute extremum values of $f$ may be either at the endpoints of the interval $[a, b]$ or at the local maximum or local minimum values of $f$ in the open interval $(a, b)$.

Remember that:

- If we have a local maximum or a local minimum value at an interior point $c$ of $[a, b]$, that is at a point $c$ of the open interval $(a, b)$, then either $f'(c)$ does not exist or $f'(c)$ exists and equals 0.
- An interior point $c$ of $[a, b]$, that is a point $c$ of the open interval $(a, b)$, is said to be a critical point of $f$ if $f'(c)$ does not exist or $f'(c)$ exists and equals 0.

So the candidates for local maximum and local minimum values at the interior points are just critical points. But note that they are just candidates, that is, at a critical point $c$, $f$ may have neither a local maximum nor a local minimum.

As a result, the absolute maximum or absolute minimum values of the continuous function $f$ on $[a, b]$ must be attained either at the endpoints of the interval $[a, b]$ or at the critical points of $f$ in the open interval $(a, b)$. So the procedure for finding the absolute maximum or absolute minimum values of $f$ is as follows:

1. Find all the critical points $c$ in the open interval $(a, b)$, that is, find all $c \in (a, b)$ such that either $f'(c)$ does not exist or $f'(c)$ exists and equals 0. So you must find for which $x \in (a, b)$, $f'(x)$ does not exist, and for which $x \in (a, b)$, $f'(x) = 0$.

2. Suppose that you have found finitely many critical points $c_1, c_2, \ldots, c_n$ of $f$ in the open interval $(a, b)$.

3. Evaluate $f$ at the endpoints $a$ and $b$ of the interval $[a, b]$ and at all of the critical points $c_1, c_2, \ldots, c_n$.

4. The absolute maximum value of $f$ is the maximum value of $\{f(a), f(b), f(c_1), f(c_2), \ldots, f(c_n)\}$.

5. The absolute minimum value of $f$ is the minimum value of $\{f(a), f(b), f(c_1), f(c_2), \ldots, f(c_n)\}$.

4 Second Derivative Test for Local Extrema

Sign Preserving Property of Continuous Functions

Remember the Sign Preserving Property of Continuous Functions:
Theorem 8. If a function \( f \) is continuous on an open interval \( I \) and if \( f(a) \neq 0 \) for a point \( a \in I \), then there exists \( \delta_0 > 0 \) such that \( f(x) \) has the same sign with \( f(a) \) for all \( x \) in the open interval \((a - \delta_0, a + \delta_0)\), that is, we have the following:

1. If \( f(a) > 0 \), then there exists \( \delta_0 > 0 \) such that \( f(x) > 0 \) for all \( x \) in the open interval \((a - \delta_0, a + \delta_0)\).

2. If \( f(a) < 0 \), then there exists \( \delta_0 > 0 \) such that \( f(x) < 0 \) for all \( x \) in the open interval \((a - \delta_0, a + \delta_0)\).

Second Derivative Test for Local Extrema

Let \( f \) be a function such that \( f'(x) \) and \( f''(x) \) exists for all \( x \) in an open interval \( I \).

Suppose that the second derivative \( f'' \) is continuous on \( I \).

If \( c \) is a critical point of \( f \) in the open interval \( I \), then \( f'(c) = 0 \) (since \( f'(x) \) exists for all \( x \in I \)).

If \( f''(c) \neq 0 \), then we can use the sign of \( f''(c) \) as stated below to determine if \( f \) has a local maximum or a local minimum value at \( c \).

If \( f''(c) = 0 \), we can say nothing: \( f \) may or may not have a local maximum or local minimum value at \( c \).

Theorem 9. Let \( f \) be a function such that \( f'(x) \) and \( f''(x) \) exists for all \( x \) in an open interval \( I \), and let \( c \in I \).

Assume that the second derivative \( f'' \) is continuous on \( I \). Then:

1. If \( f'(c) = 0 \) and \( f''(c) < 0 \), then \( f \) has a local maximum at \( c \).

2. If \( f'(c) = 0 \) and \( f''(c) > 0 \), then \( f \) has a local minimum at \( c \).

3. If \( f'(c) = 0 \) and \( f''(c) = 0 \), then the test fails, that is, \( f \) may or may not have a local maximum or local minimum value at \( c \).

See your textbook for the proof of the Second Derivative Test for Local Extrema (Section 4.4 in Chapter 4, pages 238). Note that in this proof, the Sign Preserving Property of Continuous Functions is used. If \( f''(c) \neq 0 \) and if \( f'' \) is continuous on \( I \), then by the sign preserving property of continuous functions, \( f''(x) \) will have the same sign with \( f''(c) \) for all \( x \) in an open interval \((c - \delta_0, c + \delta_0)\) for some \( \delta_0 > 0 \). Indeed, this result can be proved to hold without assuming the continuity of \( f'' \) on \( I \).

5 Application Problems

Applied Optimization

To optimize something often means to maximize or minimize some aspect of it. In the applied optimization problems, you formulate the problem mathematically and you reduce the problem usually to finding the maximum or minimum of a real-valued function \( f(x) \) of one variable \( x \) defined on interval \( I \). Determining the interval \( I \), that is, the domain of the function \( f \) comes from the application problem. Half of solving these application problems is this step of reducing the problem to finding the absolute maximum or absolute minimum value of a function \( f \) on an interval \( I \). The rest is then to use our differentiation technique for finding this absolute maximum or absolute minimum value.

See Section 4.6 (Applied Optimization) of Chapter 4 in your textbook for optimization problems (pages 255–266).
Related Rates

In related rates problems, we have for example some functions $x = x(t)$, $y = y(t)$ and $z = z(t)$ of a time variable $t$. We do not have some formula for these functions but these functions are related (for example, we may have that $x^2 + y^2 = z^2$ at all $t$) by the nature of the application problem. At some time $t_0$, we are given for example $x(t_0)$, $y(t_0)$ and $z(t_0)$, and the rate of changes of some of these functions at $t_0$, say for example $x'(t_0)$ and $y'(t_0)$ are given. Then using the relation between these functions, we are asked to find $z'(t_0)$. For example if we have the relation $x^2 + y^2 = z^2$ where $x = x(t)$, $y = y(t)$ and $z = z(t)$ are functions of $t$, then differentiating with respect to $t$ and using Chain Rule, we obtain:

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2z \frac{dz}{dt}$$

At the time $t = t_0$, we use the known given values $x(t_0)$, $y(t_0)$, $z(t_0)$, $x'(t_0)$ and $y'(t_0)$ to find $z'(t_0)$.

See Section 3.10 (Related Rates) of Chapter 3 in your textbook for related rates problems (pages 186–195).

6 Newton’s Method for finding roots of $f(x) = 0$

Roots of $f(x) = 0$

In general, it is hard to find the roots of an equation

$$f(x) = 0.$$ 

An effective numerical method to approximate a solution of an equation $f(x) = 0$ is the so called Newton’s Method which you shall also see in your Numerical Analysis I course in your 3rd year.

See Section 4.7 (Newton’s Method) of Chapter 4 in your textbook (pages 266–271) for the explanation of Newton’s Method and for applying this method to approximating roots of some equations $f(x) = 0$.

Newton’s Method for finding roots of $f(x) = 0$

Not for all functions $f$ but for some differentiable functions $f$ with nonzero derivative on an interval that contains a root of $f(x) = 0$ that have some appropriate properties, the following method gives you a sequence $(x_n)_{n=1}^{\infty}$ of real numbers that converges to that root of the equation $f(x) = 0$:

1. Guess a first approximation $x_1$ to a solution of the equation $f(x) = 0$.

2. Define recursively the sequence $(x_n)_{n=1}^{\infty}$ by the recurrence relation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \text{ for all integers } n \geq 1.$$ 

$x_{n+1}$ is the $x$-coordinate of the point where the tangent line to the graph of $y = f(x)$ at the point $(x_n, f(x_n))$ (that has slope $f'(x_n)$) crosses the $x$-axis.

The recurrence relation uses a geometric idea to approximate the roots, see Section 4.7 (Newton’s Method) of Chapter 4 in your textbook (pages 266–271).
References


