HOMEWORK 9 - TECHNIQUES OF INTEGRATION

READING:
Read the following parts from the Calculus Biographies that I have given (online supplement of our textbook):

- Thomas Simpson (1720–1761)
- Lejeune Dirichlet (1805–1859)
- Karl Weierstrass (1815–1897)

After you have studied our textbook, you may in addition use the following issues of Matematik Dünyası which contain some subjects of integration that we shall study this term:


SOLVE THE BELOW EXERCISES.

1. Evaluate the following integrals

(a) \( \int x \sin \frac{\pi}{2} \, dx \)
(b) \( \int t^2 \cos t \, dt \)
(c) \( \int e^x \ln x \, dx \)
(d) \( \int xe^{3x} \, dx \)

(e) \( \int \arctan y \, dy \)
(f) \( \int 4x \sec^2 2x \, dx \)
(g) \( \int e^y \sin \theta \, d\theta \)
(h) \( \int e^{-y} \cos y \, dy \)

(i) \( \int e^x \sin e^x \, dx \)
(j) \( \int 1/\sqrt{2} x \arcsin x^2 \, dx \)
(k) \( \int x^5 e^{x^3} \, dx \)
(l) \( \int 1 x \sqrt{1-x} \, dx \)

(m) \( \int e^x \tan^2 x \, dx \)
(n) \( \int \ln(x + x^2) \, dx \)
(o) \( \int \sin(\ln x) \, dx \)

2. Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve \( y = e^x \), and the line \( x = \ln 2 \) about the line \( y = \ln 2 \).

3. Find the volume of the solid generated by revolving the region bounded by the \( x \)-axis and the curve \( y = x \sin x \), \( 0 \leq x \leq \pi \), about (a) the \( y \)-axis, (b) the line \( x = \pi/2 \).

4. Consider the region bounded by the graphs of \( y = \tan^{-1} x \), \( y = 0 \) and \( x = 1 \). (a) Find the area of the region. (b) Find the volume of the solid formed by revolving this region about the \( y \)-axis.

5. (a) Prove the following reduction formula for all integers \( n \geq 2 \):

\[
\int \frac{1}{(1 + x^2)^n} \, dx = \frac{1}{2n-2} \left( \frac{x}{(1 + x^2)^{n-1}} + \frac{2n-3}{2n-2} \int \frac{1}{(1 + x^2)^{n-1}} \, dx \right).
\]

Hint: Start with \( \int \frac{1}{1 + x^2} = \frac{(1+x^2)-x^2}{(1+x^2)^n} = \frac{1}{(1+x^2)^n} - \frac{x^2}{(1+x^2)^n} \) and now use integration by parts for the integral \( \int \frac{x^2}{(1+x^2)^n} \, dx \).

(b) Using the above reduction formula find the indefinite integrals \( \int \frac{dx}{1 + x^2} \), \( \int \frac{dx}{(1 + x^2)^2} \) and \( \int \frac{dx}{(1 + x^2)^3} \).

6. Use integration by parts to establish the following reduction formulas:

(a) \( \int x^n \cos x \, dx = x^n \sin x - n \int x^{n-1} \sin x \, dx \)
(b) \( \int x^n \sin x \, dx = -x^n \cos x + n \int x^{n-1} \cos x \, dx \)
(c) \( \int x^n e^{ax} \, dx = \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} \, dx \)
(d) \( \int (\ln x)^n \, dx = x(\ln x)^n - n \int (\ln x)^{n-1} \, dx \).
(c) \( \int \cos^n x \, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \, dx \)

(f) \( \int \sin^n x \, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \, dx \)

7. Let \( y = f(x) \) be an invertible and differentiable function with inverse \( x = f^{-1}(y) \), use integration by parts to prove the following identity

\[
\int f^{-1}(x) \, dx = xf^{-1}(x) - \int f(y) \, dy.
\]

Then, evaluate the indefinite integrals of the functions \( \sin^{-1} x, \cos^{-1} x, \sec^{-1} x, \tan^{-1} x, \sinh^{-1} x, \) and \( \log_2 x \)

8. Evaluate the following integrals

(a) \( \int \cos^3 x \sin x \, dx \)

(b) \( \int \cos^3 x \sin^3 x \, dx \)

(c) \( \int \cos^2 x \, dx \)

(d) \( \int \frac{1}{\sqrt{9-x^2}} \, dx \)

(e) \( \int \sqrt{1-x^2} \, dx \)

(f) \( \int \frac{x}{\sqrt{1-x^2}} \, dx \)

(g) \( \int \frac{1}{\sqrt{1-x^2}} \, dx \)

(h) \( \int \frac{1}{\sqrt{1-x^2}} \, dx \)

(i) \( \int \frac{1}{\sqrt{1-x^2}} \, dx \)

(j) \( \int \frac{1}{\sqrt{1-x^2}} \, dx \)

(k) \( \int \sec^3 x \tan x \, dx \)

(l) \( \int \sec^3 x \tan^3 x \, dx \)

(m) \( \int \sec^6 x \, dx \)

(n) \( \int \tan^5 x \, dx \)

(o) \( \int \sec^6 2x \, dx \)

(p) \( \int \frac{\sin^3 \theta \cos 2\theta \, d\theta}{3} \)

9. Use trigonometric substitutions to find the following integrals

(a) \( \int \frac{dx}{\sqrt{9-x^2}} \)

(b) \( \int \frac{dx}{\sqrt{9-x^2}} \)

(c) \( \int \frac{dx}{\sqrt{9-x^2}} \)

(d) \( \int \frac{dx}{\sqrt{9-x^2}} \)

(e) \( \int \frac{dx}{\sqrt{9-x^2}} \)

(f) \( \int \frac{dx}{\sqrt{9-x^2}} \)

(g) \( \int \frac{dx}{\sqrt{9-x^2}} \)

(h) \( \int \frac{dx}{\sqrt{9-x^2}} \)

(i) \( \int \frac{dx}{\sqrt{9-x^2}} \)

10. Solve the initial value problem \( \frac{dy}{dx} = \sqrt{x^2 - 4}, \) \( x \leq 2, \) \( y(2) = 0. \)

11. Find the area enclosed by the ellipse \( \frac{x^2}{9} + \frac{y^2}{4} = 1. \)

12. Let \( a \neq 0 \) and \( a, b, c \in \mathbb{R}. \) How can you complete the quadratic form \( ax^2 + bx + c \) to the square?

13. Find the following indefinite integrals

(a) \( \int \frac{1}{x^2 + x + 1} \, dx. \) (Hint: Complete the square in \( x^2 + x + 1 \) and then make a substitution to find the integral.)

(b) \( \int \frac{1}{1+x^2} \, dx. \) (Hint: Use integration by parts, or use the trigonometric substitution \( x = \tan \theta \).)

(c) \( \int \frac{1}{1+x^2} \, dx. \)

(d) \( \int \frac{1}{x^2 + 2x + 2} \, dx. \) (Hint: Complete the square in \( x^2 + 2x + 2 \) and then make a substitution to find the integral.)

(e) \( \int \frac{1}{(x^2 + 2x + 2)^2} \, dx. \) (Hint: Firstly use a substitution (after you complete the square in \( x^2 + 2x + 2 \)) and then use integration by parts.)

(f) \( \int \frac{5x+3}{(x^2 + 2x + 2)^2} \, dx. \)

14. (a) What is the integral \( \int \frac{dx}{x^2 + a^2} \) for any real number \( a? \)

(b) For real numbers \( a, b, c \in \mathbb{R} \) such that \( b^2 - 4ac < 0, \) find the indefinite integral \( \int \frac{dx}{ax^2 + bx + c}. \)

15. For the method of integration of rational functions by partial fractions to work, we must firstly be able to factorize every polynomial with real number coefficients into a product of first and second degree polynomials with real number coefficients, where the second degree polynomials has no real roots and so cannot be expressed as a product of two first degree polynomials with real number coefficients. Is this always possible? That is, we ask if for \( n \in \mathbb{Z}^+ \) and real numbers \( a_0, a_1, a_2, \ldots, a_n \) with \( a_n \neq 0, \) does there exist \( m, r \in \mathbb{Z}^+ \cup \{0\} \) and \( x_1, x_2, \ldots, x_m \in \mathbb{R}, \) \( b_1, b_2, \ldots, b_r \in \mathbb{R} \) and \( c_1, c_2, \ldots, c_r \in \mathbb{R} \) such that

\[
a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 = a_n (x - x_1) \cdots (x - x_m) (x^2 + b_1 x + c_1) \cdots (x^2 + b_r x + c_r)
\]
where the quadratic factors \( x^2 + b_k x + c_k \) have no real roots for each \( k = 1, 2, \ldots, r \) (that is, their discriminant is negative)?

**Hint:** See the Homework on Complex Numbers.

16. In the method of integration of rational functions by partial fractions, the integration problem reduces to finding integrals of the form

\[
\int \frac{A}{(x-r)^m} \quad \text{and} \quad \int \frac{Ax + B}{(x^2 + px + q)^n}
\]

where \( n \) and \( m \) are positive integers and \( A, B, r, p, q \) are real numbers such that \( p^2 - 4q < 0 \) (so the quadratic polynomial \( x^2 + px + q \) has no real roots). Can we find these integrals in terms of our basic functions always?

17. Find the indefinite integral \( \int \frac{dx}{x^2 + x} \).

**Hint:** In the method of integration of rational functions by partial fractions, you must firstly factorize the polynomial in the denominator into a product of first degree and second degree irreducible polynomials with real number coefficients. What is the factorization of \( x^4 + 1 \)?

18. Express the integrand as a sum of partial fractions and evaluate the following integrals

(a) \( \int \frac{5x-7}{x^2-3x+2} \, dx \)
(b) \( \int \frac{x+1}{x^2-x} \, dx \)
(c) \( \int \frac{x}{x^2-x-3} \, dx \)
(d) \( \int \frac{2x^2+2}{(x+1)(x-1)^3} \, dx \)

19. Express the functions \( \sin x \), \( \cos x \), \( \tan x \), \( \sec x \), \( \csc x \) and the differential \( dx \) for the trigonometric substitution \( t = \tan \frac{x}{2} \).

20. Use trigonometric substitution \( t = \tan \frac{x}{2} \) to evaluate the following indefinite integrals

(a) \( \int \sec x \, dx \)
(b) \( \int \sec^3 \frac{x}{\tan x} \, dx \)
(c) \( \int \frac{\sin^3 x}{\cos^2 x} \, dx \)
(d) \( \int \sqrt{\tan x} \, dx \)

21. By evaluating the following improper integrals determine whether they are convergent or divergent.

(a) \( \int_{-1}^{1} \frac{1}{2} \, dx \)
(b) \( \int_{-\infty}^{\infty} \frac{dx}{1+x^2} \)
(c) \( \int_{0}^{\infty} \frac{2x \, dx}{1+e^{2x}} \)

22. For what values of \( p \) does the integral \( \int_{1}^{\infty} \frac{dx}{x^p} \) converge? When the integral does converge, what is its value?

23. State and prove the “Direct Comparison Test” for improper integrals.

24. State and prove the “Limit Comparison Test” for improper integrals.

25. For the following improper integrals, either use integratio, or the Direct Comparison Test or the Limit Comparison Test to test the integrals for convergence.

(a) \( \int_{0}^{\pi/2} \tan \theta \, d\theta \)
(b) \( \int_{0}^{\ln 2} x^{-2} e^{-1/x} \, dx \)
(c) \( \int_{0}^{\infty} \frac{dt}{e^{t} - \sin t} \)

26. Find the values of \( p \) for which each integral converges

(a) \( \int_{0}^{\pi/2} \frac{dx}{x (\ln x)^p} \)
(b) \( \int_{0}^{\infty} \frac{dx}{x (\ln x)^p} \)

27. For what value or values of \( a \) does \( \int_{1}^{\infty} \left( \frac{ax}{x^2+1} - \frac{1}{2x} \right) \, dx \) converge? Evaluate the corresponding integral(s).

28. \( \int_{-\infty}^{\infty} f(x) \, dx \) may not equal \( \lim_{b \to \infty} \int_{b}^{b} f(x) \, dx \). Show that \( \int_{0}^{\infty} \frac{2x}{x^2+1} \, dx \) diverges and hence that \( \int_{-\infty}^{\infty} \frac{2x}{x^2+1} \, dx \) diverges, while \( \lim_{b \to \infty} \int_{-b}^{b} \frac{2x}{x^2+1} \, dx = 0 \).
29. Show that if \( f(x) \) is integrable on every interval of real numbers and \( a \) and \( b \) are real numbers with \( a < b \), then

(a) \( \int_{-\infty}^{a} f(x) \, dx \) and \( \int_{b}^{\infty} f(x) \, dx \) both converge if and only if \( \int_{-\infty}^{b} f(x) \, dx \) and \( \int_{a}^{\infty} f(x) \, dx \) both converge.

(b) \( \int_{-\infty}^{a} f(x) \, dx + \int_{a}^{\infty} f(x) \, dx = \int_{-\infty}^{\infty} f(x) \, dx + \int_{b}^{\infty} f(x) \, dx \) when the integrals involved converge.

30. Consider the region in the first quadrant between \( y = -\ln x \) and the \( x \)-axis.

(a) Find the area of the region.

(b) Find the centroid of the region.

(c) Find the volume of the solid generated by revolving the region about the \( x \)-axis.

(d) Find the volume of the solid generated by revolving the region about the \( y \)-axis.

31. Euler's Gamma function \( \Gamma(x) \) uses an integral to extend the factorial function from the nonnegative integers to other real values. The formula is
\[
\Gamma(x) = \int_{0}^{\infty} t^{x-1} e^{-t} \, dt, \quad x > 0.
\]

(a) Show that this integral converges for all positive real number \( x \), i.e., \( \Gamma(x) \) is well defined on \( \mathbb{R}^+ \).

(b) Show that \( \Gamma(1) = 1 \).

(c) By applying integration by parts to the integral for \( \Gamma(x+1) \), show that \( \Gamma(x+1) = x\Gamma(x) \).

(d) Use mathematical induction and the above result to show that \( \Gamma(n+1) = n! \) for every non-negative integer \( n \).

**SOLVE THE BELOW EXERCISES FROM YOUR TEXTBOOK.**

Of course, solve as many exercises as you need to be sure that you have learned the concepts and can do computations without error but I require you to be prepared to solve some of the following exercises next week:

**Sec. 8.1 Integration by Parts (pages 436–438)**

**Sec. 8.2 Trigonometric Integrals (pages 443–444)**

**Sec. 8.3 Trigonometric Substitutions (pages 447–448)**

**Sec. 8.4 Integration of Rational Functions by Partial Fractions (pages 454–455)**

**Sec. 8.5 Improper Integrals (pages 479–481)**

**Sec. 8.6 Numerical Integration (pages 468–470)**

Study Section 8.6 independently to learn how one can calculate approximately convergent definite integrals, and learn "Trapezoidal Rule" and "Simpson's Rule" for numerical integration.

A few weeks later you have solved these exercises, to keep fresh your knowledge it is suggested to solve the following exercises from your textbook on pages 481–484:

- Questions to Guide your Review
- Practice Exercises
- Additional and Advanced Exercises