HOMEWORK 6 - INTEGRATION


READING:
Read the following parts from the Calculus Biographies that I have given (online supplement of our textbook):

1. History of the Integral
2. The Fundamental Theorem of Calculus
3. History of Differential Equations
4. Biographies of the following mathematicians (and scientists):
   • Carl Friedrich Gauss (1777-1855)
   • Georg Friedrich Bernhard Riemann (1826-1866)
   • George David Birkhoff (1884-1944)
   • Richard Dedekind (1831-1916)

After you have studied our textbook, you may in addition use the following issues of Matematik Dünyası which contain some subjects of integration that we shall study this term:

b) MD 2011-III, pages 44–77.
d) MD 2012-I, pages 21–23.
e) MD 2012-II, pages 33–36.
f) MD 2012-III, pages 27–61.

SOLVE THE BELOW EXERCISES.

1. Use finite approximations to estimate the area under the graph of the function $f(x) = 4 - x^2$ between $x = -2$ and $x = 2$, using
   a) a lower sum with four rectangles of equal width
   b) an upper sum with four rectangles of equal width

2. Evaluate the following finite sums
   a) $\sum_{k=1}^{100} k$
   b) $\sum_{k=1}^{100} k^2$
   c) $\sum_{k=1}^{100} k^3$
   d) $\sum_{k=1}^{100} k(3k + 5)$
   e) $\sum_{k=1}^{n} nk(k^2 + 3k - n)$
   f) $\sum_{k=1}^{n}(\frac{1}{n} + 2n)$

3. Find the norm of the partition $P = \{-2, -1.6, -0.5, 0, 0.8, 1\}$.

4. Express the limit $\lim_{||P|| \to \infty} \sum_{k=1}^{n}(c_k^2 - 3c_k)\Delta x_k$, where $P$ is a partition of $[-7, 5]$.

5. Inscribe a regular $n$-sided polygon inside a circle of radius $r$ and compute the area of one of the $n$ congruent triangles. Compute the area $A_n$ of the inscribed polygon. Compute the limit $\lim_{n \to \infty} A_n$, and compare the area of disk of radius $r$.

6. Let $f(x)$ be a continuous function. Express $\lim_{n \to \infty} \frac{1}{n}[f(\frac{1}{n}) + f(\frac{2}{n}) + f(\frac{3}{n}) + \cdots + f(\frac{n}{n})]$ as a definite integral.

7. Evaluate $\lim_{n \to \infty} \frac{1^5 + 2^5 + 3^5 + \cdots + n^5}{n^6}$ by showing that the limit is $\int_0^1 x^5 \, dx$.

8. Evaluate the following limits:
Graph the integrands and use areas to evaluate the following integrals:

a) \( \int_{-1}^{1} (1 - |x|) \, dx \)  

b) \( \int_{-2}^{0} \sqrt{4 - x^2} \, dx \)  

c) \( \int_{-2}^{2} (1 + \sqrt{4 - x^2}) \, dx \)  

d) \( \int_{-2}^{2} (x + \sqrt{4 - x^2}) \, dx \)

9. What values of \( \sum \) Is it true that every function \( f \) is integrable then

10. What values of \( a \) and \( b \) maximize the value of \( \int_{a}^{b} (x - x^2) \, dx \)? (Hint: Where is the integrand positive?)

11. What values of \( a \) and \( b \) minimize the value of \( \int_{a}^{b} (x^4 - 2x^2) \, dx \)?

12. Use the Max-Min inequality to find upper and lower bounds for the value of \( \int_{0}^{1} \frac{1}{1 + x^2} \, dx \).

13. Use the Max-Min inequality to show that if \( f \) is integrable then

a) \( f(x) \geq 0 \) on \( [a, b] \) \( \implies \int_{a}^{b} f(x) \, dx \geq 0 \)  

b) \( f(x) \leq 0 \) on \( [a, b] \) \( \implies \int_{a}^{b} f(x) \, dx \leq 0 \)

14. Use the inequality \( \sin x \leq x \), which holds for \( x \geq 0 \), to find an upper bound for the value of \( \int_{0}^{1} \sin x \, dx \).

15. If \( f \) really is a typical value of the integrable function \( f(x) \) on \( [a, b] \), then the constant function \( \text{av}(f) \) should have the same integral over \( [a, b] \) as \( f \). Does it? That is, does \( \int_{a}^{b} \text{av}(f) \, dx = \int_{a}^{b} f(x) \, dx \)? Give reasons for your answer.

16. It would be nice if average values of integrable functions obeyed the following rules on an interval \( [a, b] \).

a) \( \text{av}(f + g) = \text{av}(f) + \text{av}(g) \)

b) \( \text{av}(kf) = k \text{av}(f) \), where \( k \) is a real constant.

c) \( \text{av}(f) \leq \text{av}(g) \) if \( f(x) \leq g(x) \) on \( [a, b] \)

Do these rules ever hold? Give reasons for your answer.

17. Let \( f \) be a function that is differentiable on \( [a, b] \). We have defined the average rate of change of \( f \) on \( [a, b] \) to be \( \frac{f(b) - f(a)}{b - a} \), and the instantaneous rate of change of \( f \) at each \( x \in [a, b] \) to be \( f'(x) \). On the other hand, for a function \( g \) that is continuous on \( [a, b] \), we have defined the average value \( \text{av}(g) \) of \( g \) on the interval \( [a, b] \) to be \( \text{av}(g) = \frac{1}{b-a} \int_{a}^{b} g(x) \, dx \). To make consistent these two concepts, it will be reasonable to have \( \frac{f(b) - f(a)}{b - a} = \text{av}(f') \), that is to have, the average rate of change of \( f \) on \( [a, b] \) equals the average value of \( f' \) on \( [a, b] \). Is this the case? Give reasons for your answer.

18. Another motivation for the definition of the average value of a continuous function \( y = f(x) \) on an interval \( [a, b] \) is as follows. Let \( n \in \mathbb{Z}^+ \). Make a partition of \( [a, b] \) by dividing it into \( n \) intervals of length \( (b-a)/n \), that is, take the partition \( P = \{x_0, x_1, x_2, \ldots, x_{n-1}, x_n\} \) where \( x_k = a + k(b-a)/n \) for \( k = 0, 1, 2, \ldots, n \). Then \( \Delta x_k = x_k - x_{k-1} = \frac{b-a}{n} \) for all \( k = 1, 2, \ldots, n \), and so the norm of the partition \( P \) is \( ||P|| = \max\{\Delta x_1, \Delta x_2, \ldots, \Delta x_n\} = (b-a)/n \). Choose a point \( c_k \) from each interval \( [x_{k-1}, x_k] \) for \( k = 0, 1, 2, \ldots, n \). The average value of the \( n \) values \( f(c_1), f(c_2), \ldots, f(c_n) \) is \( \frac{1}{n} \sum_{k=1}^{n} f(c_k) \). What is the limit of this as \( n \to \infty \)? (Hint: Is this like a Riemann sum?)

19. Is it true that every function \( y = f(x) \) that is differentiable on \( [a, b] \) is itself the derivative of some function on \( [a, b] \)? Give reasons for your answer.

20. Use the formula \( \sum_{k=1}^{m} \sin kx = \frac{\cos \frac{b}{2} - \cos \left(\frac{m+1}{2}x\right)}{2\sin \frac{x}{2}} \) to find the area under the curve \( y = \sin x \) from \( x = 0 \) to \( x = \pi/2 \) in two steps: (a) Partition the interval \( [0, \pi/2] \) into \( n \) sub-intervals of equal length and calculate the corresponding upper sum \( U \); then (b) Find the limit of \( U \) as \( n \to \infty \) and \( \Delta x = (b-a)/n \to 0 \).

21. Evaluate the following integrals:
22. Prove the Leibniz’s Rule: if $f$ is continuous on $[a, b]$ and if $u(x)$ and $v(x)$ are both differentiable functions of $x$ whose values lie in $[a, b]$, then

$$\frac{d}{dx} \int_{u(x)}^{v(x)} f(t) \, dt = f(v(x)) \frac{dv}{dx} - f(u(x)) \frac{du}{dx}.$$ 

23. Find the derivative $\frac{dy}{dt}$ of the following functions:

a) $y = f_1 x \sin \frac{t}{t} \cos \frac{t}{t}$

b) $y = f_0 \tan x \sec^2 t \, dt$

c) $y = f_0 \tan x \sec^2 t \, dt$

d) $y = f_2 x^4 \sin x \cos t \, dt$

e) $y = f_0 e^{x^2} \sqrt{x} \, dt$

24. Use Leibniz’s Rule to find the value of $x$ that maximizes the value of the integral $\int_x^{x+3} t (5 - t) \, dt$.

25. Find $f(x)$ if $\int_1^x f(t) \, dt = x^2 - 2x + 1$.

26. Find $f(4)$ if $\int_0^x f(t) \, dt = x \cos \pi x$.

27. Suppose that $f$ has a positive derivative for all values of $x$ and that $f(1) = 0$. Which of the following statements must be true of the function $g(x) = \int_0^x f(t) \, dt$? Give reasons for your answer.

a) $g$ is a differentiable function of $x$.

b) $g$ is a continuous function of $x$.

c) $g$ has a local minimum at $x = 1$.

d) $g$ has a local maximum at $x = 1$.

e) The graph of $g$ has a horizontal tangent at $x = 1$.

28. Use substitution formula to evaluate the following indefinite integrals:

a) $\int 2x(x^2 + 5)^{-4} \, dx$

b) $\int \frac{1 + \sqrt{x}}{\sqrt{x}} \, dx$

c) $\int \sec 2t \tan 2t \, dt$

d) $\int \frac{1}{\sqrt{x^2 - 1}} \, dx$

e) $\int \tan x \sec^2 x \, dx$

29. Use substitution formula to evaluate the following indefinite integrals:

a) $\int_0^{\pi/4} \tan x \sec^2 x \, dx$

b) $\int_0^{\pi/4} (1 + e^{\tan \theta}) \sec^2 \theta \, d\theta$

c) $\int_0^{\pi/4} \frac{5r}{(3 + 2 \cos \theta)^2} \, dr$

d) $\int_0^{\pi/4} \tan \frac{x}{2} \, dx$

e) $\int_0^{\pi/4} \frac{5r}{3 + 2 \cos \theta} \, dw$

30. Sketch the following regions and find their areas:

a) the region between the curve $y = 3x^2 - 3$ and the $x$-axis, $-2 \leq x \leq 2$. 

..
b) the region between the curve \( y = x^{1/3} - x \) and the \( x \)-axis, \(-1 \leq x \leq 8\).
c) the region bounded below by the curve \( y = 1 + \cos x \) above by the line \( y = 2 \), \( 0 \leq x \leq \pi\).
d) the region bounded below by the line \( y = 1/2 \) and above by the curve \( y = \sin x \), \( \pi/6 \leq x \leq 5\pi/6\).
e) the region between the curve \( \sqrt{x} + \sqrt{y} = 1 \) and the \( x \)-axis in the first quadrant.
f) the region enclosed by the curves \( x = y^2 \) and \( x = y^3 \).
g) the region enclosed by the curves \( x = 12y^2 - 12y^3 \) and \( x = 2y^2 - 2y \).
h) the region bounded by the curve \( y = x^2 \), the line \( x + y = 2 \) and the \( x \)-axis.
i) the region enclosed by the curves \( y = \sqrt{|x|} \) and \( 5y = x + 6 \).
j) the region enclosed by the curves \( y = |x^2 - 4| \) and \( y = x^2/2 + 4 \).
k) the region enclosed by the curves \( x - y^{2/3} = 0 \) and \( x + y^4 = 2 \).
l) the enclosed by the curve \( y = \sin \frac{\pi x}{2} \) and the line \( y = x \).
m) the propeller-shaped region enclosed by the curves \( x = y^{1/3} \) and \( x = y^{1/5} \).
n) the region in the first quadrant bounded on the left by the \( y \)-axis, below by the curve \( x = \sqrt{4y} \), above left by the curve \( x = (y - 1)^2 \), and above right by the line \( x + y = 3 \).

31. Suppose \( F(x) \) is an antiderivative of \( f(x) = \sin x \), \( x > 0 \). Express \( \int_1^3 \sin \frac{2\pi x}{3} \, dx \) in terms of \( F \).

32. Show that if \( f \) is continuous, then \( \int_0^1 f(x) \, dx = \int_0^1 f(1 - x) \, dx \).

33. Let \( f \) be a continuous function on a symmetric domain \([-a, a] \). Show that:
   a) \( \int_{-a}^{a} f(x) \, dx = 2 \int_{0}^{a} f(x) \, dx \) if \( f \) is even
   b) \( \int_{-a}^{a} f(x) \, dx = 0 \) if \( f \) is odd.

34. Suppose that \( \int_0^1 f(x) \, dx = 3 \). Find \( \int_{-1}^{0} f(x) \, dx \) if \( a \) \( f \) is odd, \( b \) \( f \) is even.

35. Let \( f \) be a continuous function. Find the value of the integral \( I = \int_0^a \frac{f(x)}{f(x) + f(a-x)} \, dx \) by making the substitution \( u = a - x \) and adding the resulting integral to \( I \).

36. Prove that, if \( f \) is continuous on \([a,b]\), then \( \int_a^b f(x) \, dx = \int_{a-c}^{b-c} f(x+c) \, dx \) for any \( c \in \mathbb{R} \). This property tells us that, definite integral is invariant under translation.

37. Find the curve \( y = f(x) \) in the \( xy \)-plane that passes through the point \((9, 4)\) and whose slope at each point is \( 3\sqrt{x} \).

38. Find a curve \( y = f(x) \) passing through the point \((0, 1)\) and has a horizontal tangent line at this point, and \( y'' = 6x \) for each \( x \). How many curves like this are there? How do you know?

39. Express the solution of the initial value problem \( y' = \sec x, y(2) = 3 \) in terms of integral.

40. Solve the following initial value problems:
   a) \( y' = 4x(x^2 + 8)^{-1/3}, y(0) = 0 \).
   b) \( y' = 3 \cos^2(\frac{\pi}{4} - \theta), y(0) = \frac{\pi}{8} \).

41. Show that \( y = \int_0^x (1 + 2\sqrt{\sec t}) \, dt \) solves the initial value problem \( y'' = \sqrt{\sec x} \tan x; y'(0) = 3 \) and \( y(0) = 0 \).

42. Show that \( y = \frac{1}{a} \int_0^a f(t) \sin(a(x - t)) \, dt \) solves the initial value problem \( y'' + a^2 y = f(x), y'(0) = 0 \) and \( y(0) = 0 \).

43. Prove that \( \int_0^a \left( \int_0^u f(t) \, dt \right) \, du = \int_0^a f(u)(x-u) \, du \).

44. Evaluate the following limits:
   a) \( \lim_{b \to 1} \int_0^b \frac{dx}{\sqrt{1-x^2}} \).
   b) \( \lim_{x \to \infty} \frac{1}{x^2} \int_1^x \frac{dt}{\sqrt{t}} \).
   c) \( \lim_{x \to \infty} \frac{1}{x} \int_0^x \arctan t \, dt \).

45. Using the derivative of hyperbolic functions and inverse hyperbolic functions, prove the integration formulas given in Table 7.7 and Table 7.10 in your textbook (pages 421 and 425).
SOLVE THE BELOW EXERCISES FROM YOUR TEXTBOOK.
Of course, solve as many exercises as you need to be sure that you have learned the concepts and can do computations without error but I require you to be prepared to solve some of the following exercises next week:

Sec. 4.8 Antiderivatives (pages 277-281)
Exercises: 2, 4, 7, 8, 14–19, 21–24, 39–70, 72, 73, 76–91, 98, 104–111, 113–118, 121, 125, 127, 128.

Sec. 5.1 Area and Estimating with Finite Sums (pages 296-298)
Exercises: 1–11, 14, 21, 22.

Sec. 5.2 Sigma Notation and Limits of Finite Sums (pages 304-305)
Exercises: 1–32, 38, 43.

Sec. 5.3 The Definite Integral (pages 313-317)

Sec. 5.4 The Fundamental Theorem of Calculus (pages 325-328)
Exercises: 7–18, 21–64, 69–73, 75–84

Sec. 5.5 Indefinite Integrals and the Substitution Method (pages 333-335)

Sec. 5.5 Substitution and Area Between Curves (pages 341-344)
Exercises: 1–78, 81, 82, 85, 87, 88, 93–100, 102, 104, 106, 107, 109–118.

Sec. 7.2 Hyperbolic Functions (pages 425-428)
We have already seen hyperbolic functions and inverse hyperbolic functions, their graphs and derivatives. Just make a review of them.

Sec. 7.1 The Logarithm Defined as an Integral (pages 409-411)
Exercises: 1–9, 12–14, 23–27, 33–36, 40, 41, 47, 48, 51, 53–56, 59, 60, 64, 67, 70
Study Section 7.1 independently to learn how one can define rigorously the transcendental functions $\ln x$ and $e^x$, and obtain all of their usual properties.

A few weeks later you have solved these exercises, to keep fresh your knowledge it is suggested to solve the following exercises from your textbook on pages 345–352 and 428–430:

- Questions to Guide your Review
- Practice Exercises
- Additional and Advanced Exercises