HOMEWORK 3 - LIMITS and CONTINUITY


READING:
Read the following parts from the Calculus Biographies that I have given (online supplement of our textbook):

1. History of Limits

2. Biographies of the following mathematicians (and scientists):
   - Galileo Galilei (1564–1642)
   - Pierre de Fermat (1601–1665)
   - Bernhard Bolzano (1781–1848)
   - Augustin-Louis Cauchy (1789–1857)
   - Karl Weierstrass (1815–1897)

SOLVE THE BELOW EXERCISES.

1. For a function \( f(x) \) and \( a \in \mathbb{R} \), to find \( \lim_{x \to a} f(x) \), firstly put \( x = a \) and find \( f(a) \); if \( f(a) \) is defined, then the answer is \( \lim_{x \to a} f(x) = f(a) \). Is this a correct argument? When can this be done?

2. State the precise \( \varepsilon-\delta \) definition of \( \lim_{x \to a} f(x) = L \) where \( a, L \in \mathbb{R} \) and \( f \) is a function such that \( (a - \delta_0, a + \delta_0) \setminus \{a\} \subseteq \text{Domain}(f) \).

3. By using the precise \( \varepsilon-\delta \) definition of limit, prove the followings:
   a) \( \lim_{x \to a} c = c \), where \( c \) is a real constant
   b) \( \lim_{x \to a} x = a \)
   c) \( \lim_{x \to 5} \sqrt{x - 1} = 2 \)
   d) \( \lim_{x \to 1} (x^2 + x) = 2 \).

4. Prove that if \( \lim_{x \to a} f(x) = L \) and \( \lim_{x \to a} g(x) = M \), where \( L \) and \( M \) are real numbers, then the limits \( \lim_{x \to a} (f \mp g)(x) \) and \( \lim_{x \to a} (f \cdot g)(x) \) exist and if \( M \neq 0 \), the limit \( \frac{4}{5} \) \( f(x) \) exists, and moreover they satisfy:
   a) \( \lim_{x \to a} (f \mp g)(x) = \lim_{x \to a} f(x) \mp \lim_{x \to a} g(x) = L \pm M \)
   b) \( \lim_{x \to a} (f \cdot g)(x) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = L \cdot M \)
   c) \( \lim_{x \to a} \left( \frac{f(x)}{g(x)} \right) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L}{M} \), while \( M \neq 0 \).
   d) Assume the contrary. Can we always say that the limits \( \lim_{x \to a} f(x) \) and \( \lim_{x \to a} g(x) \) exist if one of the limits \( \lim_{x \to a} (f \mp g)(x) \) or \( \lim_{x \to a} (f \cdot g)(x) \) or \( \lim_{x \to a} \left( \frac{f(x)}{g(x)} \right) \) exist? Either prove or disprove (giving a counter example).

5. Let \( f \) be a function and \( a \) be a real number in Domain\((f)\). State the precise \( \varepsilon-\delta \) definition of continuity of \( f \) at \( a \).

6. Prove that a composition of continuous functions is a continuous function using the the precise \( \varepsilon-\delta \) definition of continuity.

7. Let \( f \) be a function such that Domain\((f)\) = \([a,b]\) for some real numbers \( a < b \). So \( f(x) \) is defined on \([a,b]\) and not defined if \( x < a \) or \( x > b \). Prove the following:
a) For a point \( c \) in the open interval \((a, b)\), \( f \) is continuous at \( c \) if and only if \( \lim_{x \to c} f(x) = f(c) \).

b) \( f \) is continuous at the left end point \( a \) of \( \text{Domain}(f) \) if and only if \( \lim_{x \to a^-} f(x) = f(a) \).

c) \( f \) is continuous at the right end point \( b \) of \( \text{Domain}(f) \) if and only if \( \lim_{x \to b^-} f(x) = f(b) \).

8. Show that if \( f \) is continuous at \( y_0 \) and \( \lim_{x \to x_0} g(x) = y_0 \) then \( \lim_{x \to x_0} (f \circ g)(x) = f(y_0) \). Under these assumptions can we say that the function \((f \circ g)\) is continuous at \( x = x_0 \)? Explain why?

9. Evaluate \( \lim_{x \to 0} e^{\sin x} \). Consider the functions \( g(x) = e^x \) and \( f(x) = \sin(x) \). Then \( e^{\sin x} = g(f(x)) \). Does the function \( f(x) \) have a removable discontinuity at \( x = 0 \)?

10. Give an example of functions \( f \) and \( g \), both continuous at \( x = 0 \), for which the composite \( f \circ g \) is discontinuous at \( x = 0 \). Does this contradict the theorem "composition of continuous functions is continuous"?

11. Give the all possible definitions of limits \( \lim_{x \to c} f(x) = \Delta \), where \( \Delta \) may be \( a, a^+, a^-, +\infty, -\infty \) and \( \Delta \) may be \( L \in \mathbb{R}, +\infty, -\infty \).

12. State and prove the Sandwich Theorem for limits as \( x \to a \) for a real number \( a \). Does the Sandwich Theorem also hold for limits as \( x \to c \) where \( c \) may be \( a^+, a^-, +\infty \) or \(-\infty \).

13. Evaluate the following limits:

   a) \( \lim_{h \to 0} \frac{5}{\sqrt{9h+4}+2} \)
   b) \( \lim_{h \to 0} \frac{\sqrt{9h+4}+2}{h} \)
   c) \( \lim_{t \to -1} \frac{t^2+3t+2}{t^2-t-2} \)
   d) \( \lim_{x \to 4} \frac{4-x}{5-\sqrt{x^2+9}} \)
   e) \( \lim_{x \to 0} \frac{1+x+\sin x}{3\cos x} \)

14. Evaluate \( \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} \) if

   a) \( f(x) = x^2 \)
   b) \( f(x) = 1/x \)
   c) \( f(x) = \sqrt{x} \).

15. Prove that \( \lim_{x \to 2} f(x) = 4 \) if \( f(x) = \begin{cases} x^2, & x \neq 2 \\ 1, & x = 2 \end{cases} \).

16. Evaluate the following limits:

   a) \( \lim_{\theta \to 3^+} \frac{|\theta|}{\theta} \)
   b) \( \lim_{t \to 4^-} (t - \lfloor t \rfloor) \)
   c) \( \lim_{x \to 0} \frac{\sin(kx)}{x} \), where \( k \) is a real constant
   d) \( \lim_{h \to 0} \frac{h}{\sin(3h)} \)
   e) \( \lim_{a \to 0} \frac{2a}{\tan a} \)
   f) \( \lim_{h \to 0} \frac{\sin(sin(h))}{\sin h} \)
   g) \( \lim_{x \to 0} \frac{\tan 3x}{\sin 8x} \)
   h) \( \lim_{x \to 0^+} f(x) \) and \( \lim_{x \to 0^-} f(x) \) if \( f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x < 0 \\ \sqrt{x}, & x \geq 0 \end{cases} \).

17. At what points the following functions are continuous?

   a) \( y = \frac{x^2+3}{x^2-3x-10} \)
   b) \( y = \frac{1}{|x+1|} \)
   c) \( y = \frac{x^2+2}{\cos x} \)
   d) \( y = \sqrt[4]{3x-1} \)
   e) \( y = (2-x)^{1/5} \)
   f) \( y = \frac{x \tan x}{x^2+1} \).

18. For what value of \( b \) the function \( g(x) = \begin{cases} x, & x < -2 \\ bx^2, & x \geq -2 \end{cases} \) is continuous on \( \mathbb{R} \)?

19. Show that the equation \( x^3 - 15x + 1 = 0 \) has three solutions in the interval \([-4, 4]\).

20. Let \( f \) be a function \( c \) be a real number \( c \) such that \((c - \delta_0, c + \delta_0) \subseteq \text{Domain}(f)\). Prove that \( f \) is continuous at \( c \) if and only if \( \lim_{h \to 0} f(c + h) = f(c) \).

22. State the Sign Preserving Property of Continuous Functions.

23. Is it true that a continuous function that is never zero on an interval never changes sign on that interval? Give reasons for your answer.

24. Given a continuous function $f$ on an interval, assume that the function $f$ has finitely many zeros on the interval, that is, the equation $f(x) = 0$ has finitely many solutions, and assume you can find all these solutions, say $a_1, a_2, \ldots, a_n$ where $a_1 < a_2 < \ldots < a_n$. Can you suggest a way to determine the sign of the function $f$ on each interval $(a_k, a_{k+1})$ for $k = 1, 2, \ldots, n - 1$ using your answers to the last three questions.

25. State the Extreme Value Theorem for Continuous Functions on a bounded closed interval.

26. State the precise $\varepsilon$-$\delta$ definition of $\lim_{x \to a^+} f(x) = L$ where $a, L \in \mathbb{R}$ and $f$ is a function such that $(a, a + \delta_0) \subseteq \text{Domain}(f)$. Using this definition, prove that $\lim_{x \to 0^+} \sqrt{x} = 0$.

27. Let $n \in \mathbb{Z}^+$ and $f(x) = x^n$. Prove the following:
   
   a) If $n$ is odd, then $f$ is a one-to-one and continuous function on $\mathbb{R}$ and its range is $\mathbb{R}$.
   
   b) If $n$ is even, then $f$ is a continuous function on $\mathbb{R}$ and its range is $[0, \infty)$ but it is not one-to-one. It will be a one-to-one function if we restrict the domain of $f$ to be $[0, \infty)$.
   
   c) You will prove in your Analysis course that the inverse of a one-to-one continuous function on an interval is also a continuous function. Using this prove that the function $g(x) = \sqrt[x]{x} = x^{1/n}$ is continuous on its domain, that is, prove that:
      
      i) If $n$ is odd, then the domain of $g$ is $\mathbb{R}$ and it is continuous on $\mathbb{R}$.
      
      ii) If $n$ is even, then the domain of $g$ is $[0, \infty)$ and it is continuous on $[0, \infty)$.
   
   d) Let $h$ be a function such that $\lim_{x \to a} h(x) = L$ where $a, L$ are real numbers.
      
      i) If $n$ is odd, prove that $\lim_{x \to a} \sqrt[n]{h(x)} = \sqrt[n]{L}$.
      
      ii) If $n$ is even and $L > 0$, prove that $\lim_{x \to a} \sqrt[n]{h(x)} = \sqrt[n]{L}$. Note that when $L > 0$, you must also prove that $h(x) > 0$ for all $x \in (a - \delta_0, a + \delta_0) \setminus \{a\}$ for some $\delta_0 > 0$; so $\sqrt[n]{h(x)}$ will be defined for all $x \in (a - \delta_0, a + \delta_0) \setminus \{a\}$ and we may consider its limit as $x \to a$.
      
      iii) If $n$ is even and $L = 0$, and if $h(x) \geq 0$ for all $x \in (a - \delta_0, a + \delta_0) \setminus \{a\}$ for some $\delta_0 > 0$, prove that $\lim_{x \to a} \sqrt[n]{h(x)} = \sqrt[n]{L} = 0$. Will this result be true if we do not assume that $h(x) \geq 0$ for all $x \in (a - \delta_0, a + \delta_0) \setminus \{a\}$ for some $\delta_0 > 0$?

28. (a) State the definition of horizontal asymptotes. When do we say that a horizontal line $y = b$ is a horizontal asymptote of a function $y = f(x)$?

   (b) How many horizontal asymptotes can a function $y = f(x)$ have?

   (c) How do we find the horizontal asymptotes of a function $y = f(x)$?

   (d) How many horizontal asymptotes can a rational function $f(x) = \frac{P(x)}{Q(x)}$ can have, where $P(x)$ and $Q(x)$ are polynomial functions? Determine all cases according to the degrees of the polynomials $P(x)$ and $Q(x)$.

29. (a) State the definition of vertical asymptotes. When do we say that a vertical line $x = a$ is a vertical asymptote of a function $y = f(x)$?

   (b) How many vertical asymptotes can a function $y = f(x)$ have?

   (c) How do we find the vertical asymptotes of a function $y = f(x)$? Can a function $y = f(x)$ have a vertical asymptote $x = a$ if $f$ is continuous at $a$? What are the candidates for vertical asymptotes?

   (d) How many horizontal asymptotes can a rational function $f(x) = \frac{P(x)}{Q(x)}$ can have, where $P(x)$ and $Q(x)$ are polynomial functions? What are the candidates for vertical asymptotes? If $a$ is a root of $Q(x)$, that is, if $Q(a) = 0$, is it necessarily true that the vertical line $x = a$ is a vertical asymptote of the rational function $f(x) = \frac{P(x)}{Q(x)}$?
30. If \( f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \) is a polynomial function of degree \( n \geq 1 \) and if \( n \) is odd, then using the Intermediate Value Theorem for Continuous functions, prove that \( f \) has at least one real number root, that is, the equation \( f(x) = 0 \) has a real number solution \( x_0: f(x_0) = 0 \).

**SOLVE THE BELOW EXERCISES FROM YOUR TEXTBOOK.**

Of course, solve as many exercises as you need to be sure that you have learned the concepts and can do computations without error but I require you to be prepared to solve some of the following exercises next week:

**Sec. 2.1 Rate of Changes and Tangents to Curves (pages 57–59)**
Exercises: 2, 4, 6, 8, 10, 14–16, 21, 22.

**Sec. 2.2 Limit of a Function and Limit Laws (pages 67–70)**

**Sec. 2.3 The Precise Definition of a Limit (pages 76–79)**
Exercises: 1, 2, 7-14, 16, 17, 20, 24, 25, 30, 33–36, 37–54, 57, 58, 60.

**Sec. 2.4 One-Sided Limits (pages 84–86)**
Exercises: 1–10, 12, 14, 15, 17, 18, 21–46.

**Sec. 2.5 Continuity (pages 95–97)**

**Sec. 2.6 Limits Involving Infinity (pages 108–113)**

A few weeks later you have solved these exercises, to keep fresh your knowledge it is suggested to solve the following exercises from your textbook on pages 110–115:

- Questions to Guide your Review
- Practice Exercises
- Additional and Advanced Exercises