COMPLEX NUMBERS. We have seen the axioms for the real number system. With the real number system at our hands, we shall construct the field of complex numbers and see some of its elementary properties.

The set \( \mathbb{C} \) of complex numbers is defined to be the set of all ordered pairs of real numbers, that is,
\[
\mathbb{C} = \mathbb{R} \times \mathbb{R} = \mathbb{R}^2 = \{(a, b) \mid a \in \mathbb{R} \text{ and } b \in \mathbb{R}\}
\]
is the 2-dimensional vector space over the field \( \mathbb{R} \) (the plane).

We will define addition of complex numbers and multiplication of complex numbers. For now, addition of complex numbers will be denoted by \( \oplus \) not to mix it with the addition of real numbers and similarly multiplication of complex numbers will be denoted by \( \odot \) not to mix it with the multiplication of real numbers.

For all \((a, b), (c, d)\) in \( \mathbb{C} = \mathbb{R} \times \mathbb{R} \), we define
\[
(a, b) \oplus (c, d) = (a + c, b + d),
\]
\[
(a, b) \odot (c, d) = (ac - bd, ad + bc).
\]

\( \mathbb{C} \) turns out to be a field with these operations, that is, it has the following properties:

- \( \oplus \) and \( \odot \) are commutative operations: For all complex numbers \( z_1 \) and \( z_2 \),
  \[
  z_1 \oplus z_2 = z_2 \oplus z_1 \quad \text{and} \quad z_1 \odot z_2 = z_2 \odot z_1.
  \]

- \( \oplus \) and \( \odot \) are associative operations: For all complex numbers \( z_1, z_2 \) and \( z_3 \),
  \[
  (z_1 \oplus z_2) \oplus z_3 = z_1 \oplus (z_2 \oplus z_3) \quad \text{and} \quad (z_1 \odot z_2) \odot z_3 = z_1 \odot (z_2 \odot z_3).
  \]

- There exist additive and multiplicative identities: For all complex numbers \( z \),
  \[
  z \oplus (0, 0) = z \quad \text{and} \quad z \odot (1, 0) = z.
  \]

- Every complex number has an additive inverse and every complex number which is not the additive identity \((0, 0)\) has a multiplicative inverse: For all complex numbers \( z = (a, b) \in \mathbb{C} = \mathbb{R} \times \mathbb{R} \), the complex number \(-z\) is defined to be \((-a, -b)\), and we have
  \[
  z \oplus (-z) = (0, 0).
  \]

For all complex numbers \( z = (a, b) \in \mathbb{C} = \mathbb{R} \times \mathbb{R} \) such that \( z \neq (0, 0) \), the complex number \( z^{-1} \) is defined to be \( \left( \frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2} \right) \), and we have
\[
 z \odot z^{-1} = (1, 0).
\]

- Multiplication distributes over addition: For all complex numbers \( z_1, z_2 \) and \( z_3 \),
  \[
  z_1 \odot (z_2 \oplus z_3) = (z_1 \odot z_2) \oplus (z_1 \odot z_3).
  \]

All of these properties are routine to check by using the corresponding property of the field \( \mathbb{R} \). We will ask you to prove only some of them.

We define
\[
i = (0, 1).
\]

Then for all \( z = (a, b) \in \mathbb{C} = \mathbb{R} \times \mathbb{R} \),
\[
i^2 = i \odot i = (-1, 0) \quad \text{and} \quad z = (a, b) = (a, 0) \oplus ((b, 0) \odot i).
\]

For the complex number \( z = (a, b) \in \mathbb{C} = \mathbb{R} \times \mathbb{R} \), the conjugate of \( z \) is defined to be \( \bar{z} = (a, -b) \) and the modulus (absolute value) of \( z \) is defined to be \( |z| = \sqrt{a^2 + b^2} \).

For all complex numbers \( z = (a, b) \) and \( w = (c, d) \), \( z \odot w \) (subtraction) is defined to be \( z \oplus (-w) \), and if \( w \neq (0, 0) \), \( \frac{z}{w} \) (division) is defined to be \( z \odot w^{-1} \).
1. Let $i = (0, 1)$.
   For all complex numbers $z = (a, b) \in \mathbb{C} = \mathbb{R} \times \mathbb{R}$, prove that:
   
   (a) $z \oplus (0, 0) = z$ and $z \odot (1, 0) = z$.
   
   (b) $z \oplus (-z) = (0, 0)$ where $-z = (-a, -b)$.
   
   (c) $z \odot z^{-1} = (1, 0)$ if $z \neq (0, 0)$, where $z^{-1} = \left( \frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2} \right)$.
   
   (d) $i \odot i = (-1, 0)$.
   
   (e) $z = (a, b) = (a, 0) \oplus ((b, 0) \odot i)$.
   
   (f) $z \odot \overline{z} = (a^2 + b^2, 0)$ and $|z|^2 = a^2 + b^2$.
   
   (g) For all $r \in \mathbb{R}$, $(r, 0) \odot (a, b) = (ra, rb)$.
   
   (h) $z^{-1} = \left( \frac{1}{a^2 + b^2}, 0 \right) \odot \overline{z}$.

2. Consider the one-to-one function $f : \mathbb{R} \rightarrow \mathbb{C}$ defined by
   
   $$f(x) = (x, 0), \quad x \in \mathbb{R}.$$ 
   
   Prove that for all real numbers $a$ and $b$,
   
   $$f(a + b) = f(a) \oplus f(b) \quad \text{and} \quad f(ab) = f(a) \odot f(b),$$
   
   that is,
   
   $$(a + b, 0) = (a, 0) \oplus (b, 0) \quad \text{and} \quad (ab, 0) = (a, 0) \odot (b, 0),$$

   (a) Using this one-to-one function, the real number $a$ is identified with $f(a) = (a, 0)$, so that the set $\mathbb{R}$ of real numbers becomes a subset of the field $\mathbb{C}$ of complex numbers. That is, the complex number $(a, 0)$ is denoted by just $a$, we take $a = (a, 0)$. In this case prove that
   
   $$(a, b) = a \oplus (b \odot i).$$

   (b) We now also change the notation $\oplus$ to just $+$ and $\odot$ to just $\cdot$. As usual, for complex numbers $z$ and $w$, the multiplication of $z$ and $w$ is just denoted by $zw$. The usual convention is that in the absence of parentheses, multiplications are performed before additions. So the previous result can be written as:
   
   $$(a, b) = a + bi.$$ 
   
   The real part of the complex number $z = (a, b) = a + bi \in \mathbb{C} = \mathbb{R} \times \mathbb{R}$ is defined to be $a$ and its imaginary part is defined to be $b$; they are denoted by $\text{Re}(z)$ and $\text{Im}(z)$ respectively.

   (c) Can we state the results of the previous questions as follows: True or false?
   
   For all complex numbers $z = (a, b) \in \mathbb{C} = \mathbb{R} \times \mathbb{R}$, that is, $z = a + bi$ where $a, b \in \mathbb{R}$:
   
   i. $z + 0 = z$ and $z \cdot 1 = z$.
   
   ii. $z + (-z) = 0$ where $-z = -a - bi$.
   
   iii. $z \cdot z^{-1} = 1$ where $z^{-1} = \frac{a}{a^2 + b^2} + \frac{-b}{a^2 + b^2}i$.
   
   iv. $i^2 = -1$.
   
   v. $\overline{z} = a - bi$.
   
   vi. $z\overline{z} = a^2 + b^2 = |z|^2$.
   
   vii. For all $r \in \mathbb{R}$, $r(a + bi) = ra + rb$.
   
   viii. $z^{-1} = \frac{\overline{z}}{z \cdot \overline{z}} = \frac{\overline{z}}{|z|^2} = \left( \frac{1}{a^2 + b^2} \right) \overline{z}$.

   (d) Prove that for real numbers $a, b, c, d$, if you multiply $(a + bi)(c + di)$ just by assuming that $i^2 = -1$ and using distributivity of multiplication over addition, commutativity and associativity of multiplication and addition, then the result is that
   
   $$(a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$
3. Polar form of complex numbers. For a complex number $z = a + ib$ where $a, b \in \mathbb{R}$, consider the polar coordinates $(r, \theta)$ of the point $(a, b)$ in the $xy$-plane where the real numbers $r$ and $\theta$ are such that

$$r = |z| = \sqrt{a^2 + b^2}, \quad a = r \cos \theta, \quad b = r \sin \theta.$$ 

Then $z = a + ib = r \cos \theta + ir \sin \theta = r(\cos \theta + i \sin \theta)$. We shall denote the part $\cos \theta + i \sin \theta$ by $e^{i\theta}$, that is, for every real number $\theta$, we define:

$$e^{i\theta} = \cos \theta + i \sin \theta.$$ 

Thus we have $z = re^{i\theta}$. This is the polar form of the complex number $z$. The angle $\theta$ is called the argument of $z$. Here $r$ is necessarily uniquely determined and when $z \neq 0$, the angle $\theta$ is determined only up to integer multiples of $2\pi$, that is, if $z \neq 0$, and for $r, s \in \mathbb{R}^+$, and $\theta, \alpha \in \mathbb{R}$,

if $z = re^{i\theta}$ and $z = se^{i\alpha}$, then $r = s$ and $\alpha = \theta + k \cdot 2\pi$ for some integer $k$.

All of these follows from trigonometry; why?

(a) For all real numbers $\theta$ and $\alpha$, prove that

$$e^{i\theta} \cdot e^{i\alpha} = e^{i(\theta + \alpha)}.$$ 

(b) De Moivre’s Theorem: For all real numbers $\theta$ and for all positive integers $n$, prove that

$$(e^{i\theta})^n = e^{in\theta},$$

that is,

$$(\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta).$$

(c) Prove the following equation involving the five mathematical constants $0, 1, \pi, e, i$: $e^{i\pi} + 1 = 0$.

(d) $n$th roots of complex numbers. Let $z = a + ib = re^{i\theta}$, where $a, b \in \mathbb{R}$, $r = |z| = \sqrt{a^2 + b^2}$ and $\theta \in \mathbb{R}$ is such that $a = r \cos \theta$ and $b = r \sin \theta$.

We want to find all $n$th roots of $z$, that is, we want to find all $w \in \mathbb{C}$ such that $w^n = z$ (so we can consider “$w = \sqrt[n]{z}$” to be an $n$th root).

If $z = 0$, then the only $n$th root of $z$ is $w = 0$.

So suppose $z \neq 0$. Thus $r \neq 0$. Suppose that $w = se^{i\alpha}$ is an $n$th root of $z$, that is, $w^n = z$, where $w = se^{i\alpha}$ is the polar form of the complex number $w$ (so $s = |w|$). Since $z \neq 0$, we cannot have $w = 0$. So $s \neq 0$. Since $r$ and $s$ are positive real numbers, we obtain:

$$w^n = z \implies |w^n| = |z| \implies |w|^n = |z| \implies s^n = r \implies s = \sqrt[n]{r}.$$ 

And then:

$$w^n = z \implies (se^{i\alpha})^n = re^{i\theta} \implies s^n e^{in\alpha} = re^{i\theta} \text{ by De Moivre’s Theorem} \implies e^{in\alpha} = e^{i\theta} \text{ by canceling } s^n = r \implies n\alpha = \theta + k \cdot 2\pi \text{ for some } k \in \mathbb{Z} \implies \alpha = \frac{\theta}{n} + k \cdot \frac{2\pi}{n} \text{ for some } k \in \mathbb{Z} \implies w = \sqrt[n]{r} e^{i \left( \frac{\theta}{n} + k \cdot \frac{2\pi}{n} \right)} \text{ for some } k \in \mathbb{Z}.$$ 

For each $k \in \mathbb{Z}$, let $w_k = \sqrt[n]{r} e^{i \left( \frac{\theta}{n} + k \cdot \frac{2\pi}{n} \right)}$.

Prove that for every $k \in \mathbb{Z}$, $w_k$ is an $n$th root of $z$ and there are no other $n$th roots of $z$ (by the above argument). Moreover, prove that because of the periodicity of the trigonometric functions $\cos$ and $\sin$, it suffices to take $n$ distinct values of $k$ modulo $n$, that is, more precisely prove that $w_0, w_1, w_2, \ldots, w_{n-1}$ are distinct and they are the only $n$th roots of $z$ in complex numbers so that $z$ has precisely $n$ distinct $n$th roots in complex numbers.
4. By constructing complex numbers, we are able to find roots of quadratic polynomial equations with real number coefficients that have no real number roots (the negative discriminant case); see Question 8. We have obtained this by adding roots $\pm i$ for the equation $x^2 + 1 = 0$. One may then ask if we need to add roots of higher degree polynomial equations. But the story ends by the complex numbers $\mathbb{C}$ because of the following very important theorem whose proof you shall see in your Complex Calculus or Algebra courses:

The Fundamental Theorem of Algebra: Let $n \in \mathbb{Z}^+$ and $a_0, a_1, a_2, \ldots, a_n$ be complex numbers with $a_n \neq 0$. Then the polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 = 0$$

has $n$ complex roots $z_1, z_2, \ldots, z_n$ counted with multiplicity (that is some of them may be equal). More precisely, we have the following factorization of the polynomial $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$ into first degree polynomials:

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 = a_n (x - z_1)(x - z_2) \cdots (x - z_n).$$

Using the Fundamental Theorem of Algebra, prove the following theorem by following the below steps:

Factorization of polynomials with real number coefficients. Let $n \in \mathbb{Z}^+$ and $a_0, a_1, a_2, \ldots, a_n$ be real numbers with $a_n \neq 0$. There exists $m, r \in \mathbb{Z}^+ \cup \{0\}$ and $x_1, x_2, \ldots, x_m \in \mathbb{R}, b_1, b_2, \ldots, b_r \in \mathbb{R}$ and $c_1, c_2, \ldots, c_r \in \mathbb{R}$ such that

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 = a_n (x - x_1) \cdots (x - x_m)(x^2 + b_1 x + c_1) \cdots (x^2 + b_r x + c_r)$$

where the quadratic factors $x^2 + b_k x + c_k$ have no real roots for each $k = 1, 2, \ldots, r$ (that is, their discriminant is negative).

As a result, every polynomial with real number coefficients has a factorization into a product of first and second degree polynomials with real number coefficients (where the second degree polynomials have no real roots and so cannot be expressed as a product of two first degree polynomials with real number coefficients).

(a) By the Fundamental Theorem of Algebra, we know that there exist complex numbers $z_1, z_2, \ldots, z_n$ such that

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 = a_n (x - z_1)(x - z_2) \cdots (x - z_n).$$

Among these roots $z_1, z_2, \ldots, z_n$, some are real and some are nonreal. Let $x_1, x_2, \ldots, x_m$ be all the real ones (where $m \in \mathbb{Z}^+ \cup \{0\}$).

(b) If a complex number $z_0$ is a root of the equation

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 = 0,$$

then prove that its conjugate $\overline{z_0}$ is also a root of this equation.

Hint: That is where we use the hypothesis that all the coefficients $a_0, a_1, a_2, \ldots, a_n$ are real numbers.

(c) So complex but nonreal roots of this equation appear in conjugate pairs $z_0$ and $\overline{z_0}$ and hence in the factorization

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 = a_n (x - z_1)(x - z_2) \cdots (x - z_n).$$

this gives us two terms $(x - z_0)(x - \overline{z_0})$. But then show that $(x - z_0)(x - \overline{z_0}) = x^2 - 2\text{Re}(z_0)x + |z_0|^2$ is then a quadratic polynomial with real number coefficients and it has no real roots since its roots are the complex but nonreal numbers $z_0$ and $\overline{z_0}$.

(d) Among the roots $z_1, z_2, \ldots, z_n$, the complex but nonreal roots are grouped into conjugate pairs and by the previous part each conjugate pair gives a quadratic polynomial $x^2 + bx + c$ that has coefficients $b, c \in \mathbb{R}$ but that has no real number roots. Let $x^2 + b_1 x + c_1, x^2 + b_2 x + c_2, \ldots, x^2 + b_r x + c_r$ be all the quadratic polynomials obtained in this way by the complex conjugate nonreal pairs of roots (where $r \in \mathbb{Z}^+ \cup \{0\}$ and where necessarily $m + 2r = n$).

(e) End the proof by combining parts (a) and (d).
5. **De Moivre’s Theorem** says that for all real numbers \( \theta \) and for all positive integers \( n \),

\[
(n \theta) = \cos(n \theta) + i \sin(n \theta).
\]

Using the binomial theorem

\[
(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k
\]

for all complex numbers \( a, b \),

\[
= \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{n-1} a b^{n-1} + \binom{n}{n} a^0 b^n
\]

one can expand the left side \((\cos \theta + i \sin \theta)^n\) and solve for \(\cos(n \theta)\) and \(\sin(n \theta)\) in terms of \(\cos \theta\) and \(\sin \theta\).

Using these for \( n = 4 \), find \(\cos(4 \theta)\) and \(\sin(4 \theta)\) in terms of \(\cos \theta\) and \(\sin \theta\).

6. Find the four distinct complex numbers \( z \) such that \( z^4 + 1 = 0 \).

7. Prove that every complex number has a square root by directly finding a formula, that is, prove that for every complex number \( z = a + bi \) where \( a, b \in \mathbb{R} \), there exits a complex number \( w = x + yi \) where \( x, y \in \mathbb{R} \) such that \( w^2 = z \), i.e., \((x + yi)^2 = a + bi\), by directly finding a formula for \( x \) and \( y \) in terms of \( a \) and \( b \).

Indeed, using the polar form of a complex number, we have already seen that every complex number has an \( n \)th root for every positive integer \( n \). Here what we ask is a formula for \( x \) and \( y \) in terms of \( a \) and \( b \). You can either try to solve directly for \( x \) and \( y \) in terms of \( a \) and \( b \) or you can use polar form but in the end give a formula for \( x \) and \( y \) just in terms of \( a \) and \( b \).

8. Let \( a, b, c \in \mathbb{R} \) and \( a \neq 0 \). Consider the quadratic equation

\[
ax^2 + bx + c = 0.
\]

The **discriminant** of the quadratic polynomial \( ax^2 + bx + c \) is defined to be the real number

\[
\Delta = b^2 - 4ac.
\]

Obtain the **quadratic formula** for the roots of the equation by using the method of completing the square. More precisely prove the following:

(a) There exist unique **complex numbers** \( z_1 \) and \( z_2 \) such that we have the following factorization of the quadratic polynomial \( ax^2 + bx + c \) into first degree polynomials:

\[
ax^2 + bx + c = a(x - z_1)(x - z_2)
\]

(b) Depending on the sign of \( \Delta = b^2 - 4ac \), there exists three cases:

i. If \( \Delta > 0 \), then \( z_1 \) and \( z_2 \) are real numbers and \( z_1 \neq z_2 \).

In this case, the roots are the distinct real numbers

\[
z_1 = \frac{-b + \sqrt{\Delta}}{2a} \quad \text{and} \quad z_2 = \frac{-b - \sqrt{\Delta}}{2a}.
\]

ii. If \( \Delta = 0 \), then \( z_1 \) and \( z_2 \) are real numbers and \( z_1 = z_2 \).

In this case, the roots are the same real number

\[
z_1 = z_2 = \frac{-b}{2a}.
\]

iii. If \( \Delta < 0 \), then \( z_1 \) and \( z_2 \) are complex numbers but not real numbers, and \( z_1 \neq z_2 \).

In this case, the roots are the distinct complex numbers

\[
z_1 = \frac{-b + i\sqrt{|\Delta|}}{2a} \quad \text{and} \quad z_2 = \frac{-b - i\sqrt{|\Delta|}}{2a}.
\]

They are not real numbers, so we have no real roots in the case \( \Delta < 0 \).
(c) Then observe that the quadratic polynomial \( ax^2 + bx + c \) always has two roots \( z_1 \) and \( z_2 \) in complex numbers (counted with multiplicity, that is, we may have \( z_1 = z_2 \)) but the ‘discriminant’ \( \Delta = b^2 - 4ac \) discriminates the cases whether these two roots are equal or distinct. That is why it is called the discriminant. If \( \Delta = 0 \), then we have repeated roots. If \( \Delta \neq 0 \), then we have either two distinct real number roots (if \( \Delta > 0 \)) or two distinct complex number roots (if \( \Delta < 0 \)).

(d) We can then say that by a correct interpretation of \( \sqrt{\Delta} \), we always have the following quadratic formula for the roots \( z_1 \) and \( z_2 \) in complex numbers of the quadratic equation \( ax^2 + bx + c = 0 \) where \( a, b, c \in \mathbb{R} \) with \( a \neq 0 \):

\[
z_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a}, \quad \text{the discriminant} \quad \Delta = b^2 - 4ac.
\]

In this formula, if \( \Delta < 0 \), then we think \( \sqrt{\Delta} \) to be the complex number \( i\sqrt{|\Delta|} \) (whose square is then \( \Delta \) since \( \Delta < 0 \)).

Note that for a nonnegative real number \( r \), we denote by \( \sqrt{r} \), the unique nonnegative real number whose square is \( r \).