1. Find the following limits:

(a) \[ \lim_{x \to \infty} \frac{13x^{3/2} + 21x - \ln x}{\sqrt{8x^3} + 5x + 13} \]
(b) \( \lim_{x \to 0^+} x^{1/5} \ln x \)

(c) \( \lim_{x \to \infty} \left( 1 + \frac{1}{x^2} \right)^x \)
2. Consider the function

\[ f(x) = x^{1/3}, \quad -\infty < x < \infty. \]

Explain and prove your answers to the following questions for the function \( f \):

(a) Is \( f \) a **continuous** function?

(b) Is \( f \) **differentiable** at 0? If so, what is \( f'(0) \)?

**Hint:** Use the definition of the derivative.

(c) Using the derivative rule for power functions, find \( f'(x) \) for all \( x \neq 0 \), and then find

\[ \lim_{x \to 0} f'(x). \]
3. (a) Using the definition of the derivative, prove the **Quotient Rule for Derivatives**:

If $f(x)$ and $g(x)$ are differentiable functions, then prove that $\frac{f(x)}{g(x)}$ is also a differentiable function and

$$
\frac{d}{dx} \left( \frac{f(x)}{g(x)} \right) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.
$$

(b) Find the derivative of the tangent function $\tan x = \frac{\sin x}{\cos x}$ by using the Quotient Rule for Derivatives. You can use without proving that $\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(\cos x) = -\sin x$. 
(c) Is the tangent function \( \tan x \) one-to-one on the interval \((-\pi/2, \pi/2)\)?

If so, find the derivative of its inverse function \( \tan^{-1} : \mathbb{R} \rightarrow (-\pi/2, \pi/2) \) using the rule for the derivative of the inverse function: \((f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}\).

**Hint:** What is the sign of the derivative of \( \tan x \) on \((-\pi/2, \pi/2)\)? Is the function \( \tan x \) increasing/decreasing on \((-\pi/2, \pi/2)\)?
4. (a) Let $f(x)$ be a function that is **differentiable** on $\mathbb{R}$. If the derivative $f'(x) \neq 0$ for all $x \in \mathbb{R}$, then prove that $f(x)$ can have **at most one root** in $\mathbb{R}$.

**Hint:** What happens if $x_1$ and $x_2$ are two distinct zeros of $f(x)$, say

$$ x_1 < x_2 \quad \text{and} \quad f(x_1) = f(x_2) = 0. $$

(b) Prove that the cubic polynomial function $f(x) = x^3 + 5x^2 + 10x - 7$ has **exactly one** real root.

**Hint:** Use the **Intermediate Value Theorem for Continuous Functions** to prove that it has a real number root. Then use part (a) to prove that it has at most one real number root.
5. Suppose that for all $x$ in an open interval containing $\pi/4$, the equation $xy^3 + \tan(x + y) = 1$ has a unique solution for $y$ in $\mathbb{R}$ which we denote by $y = f(x)$. Further assume that $y = f(x)$ is a differentiable function of $x$. Thus we have a differentiable function $y = f(x)$ that has been implicitly defined for all $x$ in an open interval containing $\pi/4$ such that

$$xy^3 + \tan(x + y) = 1.$$ 

(a) Find $y' = \frac{dy}{dx}$ in terms of $x$ and $y$.

(b) Find the equation of the tangent line at the point $(\pi/4, 0)$ on the curve defined by the equation $xy^3 + \tan(x + y) = 1$. 
6. Among all right circular cones with a slant height of 3, what are the dimensions (radius and height) that MAXIMIZE the volume of the cone? The slant height of a cone is the distance from the outer edge of the base to the vertex of the cone; see the figure.
Consider the **hyperbolic sine** function

\[
\sinh x = \frac{e^x - e^{-x}}{2}.
\]

(a) Prove that \(\sinh x\) is a one-to-one function by showing that it is an **increasing function** on \(\mathbb{R}\).

**Hint:** Find the derivative of \(\sinh x\) just by using the fact that \(\frac{d}{dx}(e^x) = e^x\) (and chain rule of course). What is the sign of \(\frac{d}{dx}(\sinh x)\)?

(b) Prove that the **inverse** of the one-to-one function \(\sinh\) is given by

\[
\sinh^{-1} x = \ln \left( x + \sqrt{x^2 + 1} \right), \quad -\infty < x < \infty.
\]
8. Draw the graph of the rational function

\[ f(x) = \frac{10x^3}{x^2 - 1} \]

by following the graphing strategy.

**GRAPHING STRATEGY:**

(a) Identify the domain of \( f \) and any symmetries the curve may have.

(b) Find \( f' \) and \( f'' \).

(c) Find any critical points of \( f \) and identify the function’s behavior at each one.

(d) Find where the curve is increasing and where it is decreasing.

(e) Find the points of inflection, if any occur, and determine the concavity of the curve.

(f) Identify any asymptotes (horizontal asymptotes and vertical asymptotes).

(g) Plot key points, such as the intercepts and the points found in the steps (c), (d) and (e), and plot the asymptotes if any.

(h) Gathering all you have found in the previous steps, sketch the curve.