Calculus Biographies

Niels Henrik Abel
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Carl Friedrich Gauss
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James Gregory
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Christiaan Huygens
Carl Gustav Jacob Jacobi
Johannes Kepler
Sonya Kovalevsky
Joseph Louis Lagrange
Pierre-Simon Laplace
Adrien-Marie Legendre
Colin Maclaurin
Marin Mersenne
John Napier
Isaac Newton
Nicole Oresme
Blaise Pascal
Jules-Henri Poincaré
Siméon-Denis Poisson
Bernhard Riemann
Michel Rolle
Carl Runge
Thomas Simpson
Willebrord Snell
Brook Taylor
Evangelista Torricelli
Gregory Saint Vincent
Karl Weierstrass
History of Limits

Limits present us with a grand paradox. Every major concept of calculus – derivative, continuity, integral, convergence/divergence – is defined in terms of limits. Limit is the most fundamental concept of calculus; in fact, limit is what distinguishes at the most basic level what we call calculus from algebra, geometry, and the rest of mathematics. Therefore, in terms of the orderly and logical development of calculus, limits must come first.

But the historical record is just the reverse. For many centuries, the notions of limit were confused with vague and sometimes philosophical ideas of infinity (infinitely large and infinitely small numbers and other mathematical entities) and with subjective and undefined geometric intuitions. The term limit in our modern sense is a product of the late 18th and early 19th century Enlightenment in Europe, and our modern definition is less than 150 years old. Up until this time, there were only rare instances in which the idea of the limit was used rigorously and correctly.

The first time limits were needed was for the resolution of the four paradoxes of Zeno (ca. 450 B.C.). In the first paradox, the Dichotomy, Zeno pits an object moving a finite distance between two fixed points against an infinite series of time intervals (the time it takes to move half the distance, then the time it takes to move half the remaining distance, etc.) during which the motion is supposed to take place. Zeno’s striking conclusion was that motion is impossible! Aristotle (384–322 B.C.) attempted to refute Zeno’s paradoxes with philosophical arguments. In mathematics, careful application of the limit will resolve the questions raised by Zeno’s paradoxes.

For his rigorous proofs of formulas for certain areas and volumes, Archimedes (287--212 B.C.) encountered several infinite series – sums that contain an infinite number of terms. Lacking the concept of the limit per se, Archimedes devised very ingenious arguments called double reductio ad absurdum that actually incorporated some of the technical details of what we now call limits.

Calculus is also sometimes described as the study of curves, surfaces, and solids. The development of the geometry of these objects blossomed following the invention of analytic geometry by Pierre Fermat (1601-1665) and René Descartes (1596--1650). Analytic geometry is essentially the marriage of geometry and algebra, and each one enhances the other.

Fermat devised an algebraic method, which he called adequality, for finding the highest and lowest points on certain curves. Describing the curve in question by an equation, Fermat let E stand for a small number, then he did some legitimate algebraic calculations, and finally assumed E = 0 so that all of the remaining terms in which E appeared would vanish! Essentially, Fermat sidestepped the limit with the argument that E is "infinitesimally small." Geometrically, Fermat was attempting to show that exactly at the highest and the lowest points along a curve, the straight lines tangent to that curve are flat, i.e., have slope zero.

Finding straight lines tangent to curves turns out to be one of the two most fundamental problems of calculus. Tangent problems are a part of what we now call the study of derivatives. During the seventeenth century, several geometers devised complicated algebraic schemes for finding tangent lines to certain curves. Descartes had a process that used double-roots of an auxiliary equation, and this technique was improved upon by the mathematician Johan Hudde (1628--1704), who was also the mayor of Amsterdam. René de Sluse (1622--1685) invented another even more involved method of obtaining tangent lines to curves. In each of these computations, the limit should have been used at some critical step, but wasn’t. None of these geometers perceived the need for the limit idea, and so each one found a clever way to achieve his results, which were correct, but by means that we now recognize lack rigorous foundations.

Determining exact values for areas of regions bounded at least in part by curves is the second fundamental problem of calculus. These are often called quadrature problems, and closely related to them are cubature problems -- finding volumes of solids bounded at least in part by curved surfaces. They all lead to integrals. Johannes Kepler (1571--1630), the famous astronomer, was one of the earliest students of cubature problems. Bonaventura Cavalieri (1598--1647) developed an elaborate theory of quadratures. Others, such as Evangelista Torricelli (1608--1647), Fermat, John Wallis (1616--1703), Gilles Personne de Roberval (1602--1675), and Gregory St. Vincent (1584--1667) devised quadrature and/or cubature techniques that applied to specific curves and solids or families of curves. But none used limits! Their results were nearly all correct, but each depended on an algebraic sleight-of-hand or an appeal to questionable geometric or philosophical intuition at some critical point. The need for limits was just not recognized.
In almost all of his work that is now considered calculus, *Isaac Newton* (1642--1727), also failed to acknowledge the fundamental role of the limit. For infinite series, Newton merely reasoned by analogy: if one could perform algebraic operations on polynomials, then one ought to be able to do the same with the infinite number of terms of an infinite series. Newton calculated what he called *fluxions* to curves, not quite derivatives, but very close. The process he used for these computations was very similar to *Fermat's* adequality. In these and most other comparable work, Newton overlooked the limit.

On the other hand, in his *Principia Mathematica* (1687), arguably the greatest work ever in mathematics and science, Newton was the first to recognize that the limit must be the starting point to tangent, quadrature, and related problems. At the very beginning of Book I of the *Principia*, Newton attempted to give a precise formulation of the *limit* concept:

> Quantities, and the ratios of quantities, which in any finite time converge continually to equality, and before the end of that time approach nearer to each other than by any given difference, become ultimately equal.

There were criticisms made of this statement and of the argument that followed it, most notably by *George Berkeley* (1685--1753). But the genius of Newton had discovered the primary role that the limit had to play in the logical development of calculus. And, as convoluted as his language was, the seed of the modern definition of *limit* was present in his statements.

Unfortunately, for the rigorous foundation of calculus, for many decades, no one took up these hints that Newton had provided. The major contributions to calculus of *Gottfried Wilhelm Leibniz* (1646--1716) were the notations and basic formulas for derivatives and integrals (which we have used ever since) and the *Fundamental Theorem of Calculus*. With these powerful tools, the number of curves and solids for which derivatives and integrals could be easily calculated expanded rapidly. Challenging geometry problems were solved; and more and more applications of calculus to science, mainly physics and astronomy, were discovered; and whole new fields of mathematics, especially differential equations and the calculus of variations, were created. Among the leaders of these 18th century developments were several members of the Bernoulli family, *Johann I* (1667--1748), *Nicolas I* (1687--1759) and *Daniel* (1700--1782), *Brook Taylor* (1685--1731), *Leonhard Euler* (1707--1783), and *Alexis Claude Clairaut* (1713--1765).

Calculus was swept along by its many successes in the 18th century, and very little attention was paid to its foundations, much less to the limit and its details. *Colin Maclaurin* (1698--1746) defended Newton’s treatment of fluxions against the attacks of George Berkeley. But Maclaurin reverted to 17th century arguments similar to Fermat’s adequality and only occasionally used Archimedes’ double *reductio ad absurdum*. Despite his good intentions, Maclaurin passed over opportunities to pursue Newton’s suggestion about limits. *Jean Le Rond d’Alembert* (1717--1783) was the only scientist of the time to explicitly recognize the centrality to calculus of the limit. In the famous *Encyclopédie* (1751--1776), d’Alembert asserted that the proper definition of the derivative required understanding the limit first, and then he gave the explicit definition:

> One magnitude is said to be the *limit* of another magnitude when the second may approach the first within any given magnitude, however small, though the second magnitude may never exceed the magnitude it approaches.

In general terms, d’Alembert perceived that, "the theory of limits was the true metaphysics of *calculus*."

Concern about the lack of rigorous foundations for calculus grew during the late years of the 18th century. In 1784, the Berlin Academy of Science offered a prize for an essay that would successfully explain a theory of the infinitely small and the infinitely large in mathematics and that could, in turn, be used to put calculus on a firm logical footing. Though this prize was awarded, the "long and tedious" winning work by *Simon L’Huilier* (1750--1840) was not considered a workable solution of the problems posed. *Lazare N. M. Carnot* (1753--1823) composed a popular attempt to explain the role of the limit in calculus as "the compensation of errors" – but he did not explain how these errors would always balance each other perfectly.

By the end of the 18th century, the greatest mathematician of the time, *Joseph-Louis Lagrange* (1736--1813), had succeeded in reformulating all of mechanics in terms of calculus. In the years immediately following the French Revolution, Lagrange focused the powers of his attention on the problems of the foundations of calculus. His solution, *Analytic Functions* (1797), was to "detach" calculus from "any consideration of infinitely small or evanescent quantities, of limits or of *fluxions*." Renowned for his other contributions to calculus, Lagrange made a heroic effort (as we know now, fatally flawed) to make calculus
Throughout the 18th century, there was very little concern given to the convergence or divergence of infinite sequences and series; today, we understand that such problems require the use of limits. In 1812, Carl Friedrich Gauss (1777--1855) composed the first strictly rigorous treatment of the convergence of sequences and series, although he did not use the terminology of limits. In his famous Analytic Theory of Heat, Jean Baptiste Joseph Fourier (1768--1830) tried to define the convergence of an infinite series, again without using limits; but then he claimed that any function whatsoever could be written as one of his series, and he made no mention of the convergence or divergence of this series.

In the first careful and rigorous study of the distinctions between continuous and discontinuous curves and functions, Bernhard Bolzano (1781--1848) looked beyond the intuitive notion of the absence of holes and breaks and found the more fundamental concepts which we express today in terms of limits.

At the beginning of the 18th century, the ideas about limits were certainly confusing. While Augustin Louis Cauchy (1789-1857) was searching for a clear and rigorously correct exposition of calculus to present to his engineering students at the École polytechnique in Paris, he found errors in the program set forth by Lagrange. Thus, Cauchy started his calculus course from scratch; he began with a modern definition of the limit. Beginning in 1821, he began writing his own class notes, essentially his own textbooks, the first one called Cours d’analyse (Course of Analysis). In his classes and in these classic textbooks, Cauchy used the limit principle as the basis for precise introductions to continuity and convergence, the derivative, the integral, and the rest of calculus.

However, Cauchy had missed some of the technical details, especially in the application of his definition of the limit to continuous functions and the convergence of certain infinite series. Niels Henrik Abel (1802--1829) and Peter Gustav Lejeune Dirichlet (1805--1859) were among those who ferreted out these delicate and non-intuitive problems. In the 1840s and 1850s, while he was a high school teacher, Karl Weierstrass (1815--1897) determined that the first step required to correct these errors was to restate Cauchy’s original definition of the limit in strict arithmetical terms, using only absolute values and inequalities. Weierstrass’ exposition is exactly that found in Thomas’ Calculus. Weierstrass went on to an illustrious career as a professor of mathematics at the University of Berlin. There he developed a path-breaking program designed to bring arithmetical rigor to all of calculus and mathematical analysis.
History of the Derivative

The derivative has two basic facets, the geometric and the computational. In addition, the applications of the derivative are manifold: the derivative plays many important roles in mathematics itself; it has uses in physics, chemistry, engineering, technology, sciences, economics, and much more, and new applications are devised every day.

The origin of the derivative resides in the classical geometric tangent problems, e.g., to determine a straight line that intersects a given curve in only one given point. Euclid (ca. 300 B.C.) proved the familiar high school geometry theorem to the effect that the line tangent to a circle at any point P is perpendicular to the radius at P. Archimedes (287--212 B.C.) had a procedure for finding the tangent to his spiral. And Apollonius (ca. 262--190 B.C.) described methods, all somewhat different, for determining tangents to parabolas, ellipses, and hyperbolas. But these were just geometric problems that were studied for their own very limited interests; the ancient Greeks did not perceive any common thread or other value to these theorems.

Problems of motion and velocity, also basic to our understanding of the derivative today, also originated with the ancient Greeks, although these questions were originally treated more philosophically than mathematically. The four paradoxes of Zeno (ca. 450 B.C.) hinge on difficulties with understanding instantaneous velocity without having a grasp of the derivative. In the Physics of Aristotle (384--322 B.C.), motion problems are closely associated with ideas of continuity and the infinite (i.e. infinitely small and infinitely large magnitudes). In medieval times, Thomas Bradwardine (1295--1349) and his colleagues at Merton College, Oxford, made the first efforts to transform some of Aristotle’s ideas about motion into quantitative statements. In particular, the notion of instantaneous velocity became measurable, at least in theory; today, it is the derivative (or rate of change) of the distance with respect to the time.

It was Galileo Galilei (1564--1642) who established the principle that mathematics was the indispensable tool for studying motion and, in general, science: "Philosophy [science and nature] is written in that great book which ever lies before our eyes – I mean the universe – but we cannot understand it if we do not first learn the language ... The book is written in the mathematical language ...." Galileo studied motion geometrically; he used the classical proportionalities of Euclid and properties of the conics from Apollonius to derive relationships among distance, velocity, and acceleration. Today, these variable quantities are basic applications of the derivative.

Interest in tangents to curves reappeared in the 17th century as a part of the development of analytic geometry. Because equations were then used to describe curves, the number and variety of curves increased tremendously over those studied in classical times. For instance, Pierre Fermat (1601--1665) was the first to consider the idea of a whole family of curves at once. He called these higher parabolas, curves of the form, \( y = kx^n \), where \( k \) is constant and \( n = 2, 3, 4, ... \). The introduction of algebraic symbols as a tool for studying the geometry of curves contributed significantly to the development of the derivative, the integral, and calculus. On the other hand, because correct geometric results and conclusions could be attained more easily using algebra than by geometric reasoning, the standards of logical rigor that had been pioneered by the ancient Greeks were relaxed in many calculus problems, and this (among other factors) led to some spirited and even bitter controversies. Fermat devised an algebraic procedure that he called adequality to determine the highest (maximum) and the lowest (minimum) point(s) on a curve; geometrically, he was finding the point(s) where the tangent to the curve had slope zero.

René Descartes (1596--1650) had the insight to foresee the importance of the tangent when, in his Geometry, he wrote, "And I dare say that this [finding the normal, or perpendicular to a curve, from which we can easily find the tangent] is not only the most useful and general problem in geometry that I know, but even that I have ever desired to know." Descartes devised a double root procedure for finding the normal and thus the tangent to a curve. As a result of the translation of Descartes’ Geometry into Latin by Frans van Schooten (1615--1661) and the extensive explanations by van Schooten, Florimonde de Beaune (1601--1652) and Johan Hudde (1628-1704), the principles and benefits of analytic geometry became more widely known. In particular, Hudde simplified Descartes’ double root technique to determine maximum and minimum points on a curve; the double root procedure was rediscovered by Christiana Huygens (1629-1695). Then, by modifying Fermat’s tangent process, Huygens devised a sequence of algebraic steps that produced the inflection point(s) of a curve; we will see that this requires the second derivative. René François de Sluse (1622--1685) derived an algebraic technique that led to the slope of the tangent to a curve. In the late 1650s, there was a good deal of correspondence between Huygens, Hudde, van Schooten, Sluse, and others concerning tangents of many algebraic curves; Hudde and Sluse especially sought simpler and standardized algebraic methods that could be applied to a greater variety of curves. For Gilles Personne de Roberval (1602--1675), a curve was the path of a moving point, and he...
derived a mechanical method for finding the tangent to many curves, including the cycloid. But Roberval’s method could not be generalized to include more curves.

Isaac Newton (1642--1727) began developing his "fluxional calculus," among his earliest scientific endeavors in 1663. For Newton, motion was the "fundamental basis" for curves, tangents, and related calculus phenomena, and he derived his fluxions from Hudde’s version of the double root procedure. Newton extended this technique as a method for finding the curvature of a curve, a feature that we now know is an application of the second derivative. In 1666, 1669, and 1671, Newton summarized and revised his calculus work, and these manuscripts were circulated among a good number of his colleagues and friends. Yet, though he continued to return to calculus problems at various times later in his scientific life, Newton’s calculus works were not published until 1736 and 1745.

With some tutoring and advice from Huygens and others, Gottfried Wilhelm Leibniz (1646--1716) developed his differential and integral calculus during the years 1673--1676 while he was living as a diplomat in Paris. On a short trip to London, where he attended a meeting of the Royal Society in 1673, Leibniz learned of Sluse’s method of finding tangents to algebraic curves. Leibniz had little inclination to derive these rules and even less interest in the mathematical foundations (i.e., limits) required, but he did perfect the modern formulas and notation for the derivative in his famous paper, "New methods for maximums and minimums, as well as tangents, which is neither impeded by fractional nor irrational quantities, and a remarkable calculus for them." (1684)

Here is the first published work in calculus and in fact the first use of the word "calculus" in the modern sense. Now, anyone could solve tangent problems without mastering geometry, one could simply use the "calculus" of Leibniz’s formulas.

Newton and Leibniz are sometimes said to have "invented" the calculus. As we can see, that is very much of an oversimplification. Rather, as Richard Courant (1888--1972) wisely observed, calculus has been "a dramatic intellectual struggle which has lasted for 2500 years." After 1700, circumstances led to one of the saddest and most disgraceful episodes in all of the history of science: the priority dispute between Leibniz and Newton, and even more between their followers, over who should receive credit for the calculus. Each made major contributions to the derivative, the integral, infinite series, and most of all to the Fundamental Theorem of Calculus. The charges of plagiarism and other ugly attacks were irrelevant to the mathematics done by each man, but the accusations and counter-slurs escalated into schisms between mathematicians and scientists in England (loyal to Newton) and on the European continent (followers of Leibniz) which for more than a century even led to nationalistic xenophobia.

The first book on the differential calculus was Analysis of Infinitely Small Quantities for the Understanding of Curved Lines (1696) by the Marquis de l’Hospital (1661--1704). Much of this work was actually due to Johann Bernoulli (1667--1748) and followed Leibniz’s treatment of derivatives, maximums, minimums and other analyses of curves. But l’Hospital’s method of determining the radius of curvature was very similar to that of Newton. Jakob Bernoulli (1654-1705) and his younger brother Johann led the way in spreading the word about the power of Leibniz’s calculus formulas by proposing and solving several challenging problems (the catenary problem and the brachistochrone problem are two examples) for which calculus was required. Leibniz, Newton, and Huygens also solved these problems. These problems and others led to the development of differential equations and calculus of variations, whole new fields of mathematics dependent upon calculus.

In England, Thomas Simpson’s (1710--1761) New Treatise of Fluxions (1737) provided the first derivative of the sine function. In 1734, Bishop George Berkeley (1685--1753) published The Analyst, a scathing attack on Newton’s lack of rigorous foundations for his fluxions. Berkeley acknowledged the accuracy of Newton’s formulas and correctness of their far-reaching applications in physics and astronomy, but he criticized the "infinitely small quantities" and the "evanescent increments" of the foundations of the derivative. Colin Maclaurin (1698--1746) tried to defend Newton in his Treatise of Fluxions (1742), and he derived derivatives for logarithms and exponentials and expanded Simpson’s formulas to include the derivatives of the tangent and the secant.

On the continent, Maria Agnesi (1718--1799) followed Leibniz and L’Hospital in her calculus book, Analytical Institutions (1748). Leonhard Euler (1707--1783) took a major step toward establishing a firmer foundation for calculus in his Introduction to the Analysis of the Infinite (1748) when he introduced functions (as opposed to curves) as the objects to which the derivative and the other techniques of the calculus would be applied. By function, Euler meant some kind of an "analytic expression;" his conception was not so broad as our modern definition. In this publication, he also introduced the term analysis as a modern name for calculus and related higher mathematics. In his Methods of Differential Calculus (1755), Euler defined the derivative as "the method of determining the ratios of the evanescent increments which functions receive to those of the evanescent increments of the variable quantities, of which they are the functions," which sounds very unscientific today. Even so, Euler dealt with several special cases of the
chain rule, he introduced differential equations, and he treated maximums and minimums without using any diagrams or graphs. In 1754, in the famous French Encyclopédie, Jean le Rond d’Alembert (1717--1783) asserted that the "most precise and neatest possible definition of the differential calculus" is that the derivative is the limit of certain ratios as the numerators and denominators get closer and closer to zero, and that this limit produces certain algebraic expressions that we call the derivative.

At the end of the 18th century, Joseph Louis Lagrange (1736--1813) attempted to reform calculus and make it rigorous in his Theory of Analytic Functions (1797). Lagrange intended to give a purely algebraic form for the derivative, without any recourse to geometric intuition or graphs or diagrams and without any help from d’Alembert’s limits. Lagrange did invent the prime notation we now use for derivatives, and the logical development of his calculus was admirable in other respects, but his effort to provide a firm foundation for the calculus failed because his conception of the derivative was based on certain properties of infinite series which we now know are not true.

Finally, in the early years of the 19th century, the modern definition of the derivative was given by Augustin Louis Cauchy (1789--1857) in his lectures to his engineering students. In his Résumé of Lessons given at l’Ecole Polytechnique in the Infinitesimal Calculus (1823), Cauchy stated that the derivative is:

\[
\text{the limit of } \frac{f(x + i) - f(x)}{i} \text{ as } i \text{ approaches 0. The form of the function which serves as the limit of the ratio } \frac{f(x + i) - f(x)}{i} \text{ will depend on the form of the proposed function } y = f(x). \text{ In order to indicate this dependence, one gives the new function the name of derived function.}
\]

Cauchy went on to find derivatives of all the elementary functions and to give the chain rule. Of equal importance, Cauchy showed that the Mean Value Theorem for derivatives, which had appeared in Lagrange’s work, was actually the cornerstone for proving a number of basic calculus theorems that had been taken for granted, e.g., descriptions of increasing and decreasing functions. Derivatives and the differential calculus were now established as a rigorous and modern part of calculus.
Integral calculus originated with quadrature and cubature problems. To solve a quadrature problem means to find the exact value of the area of a two-dimensional region whose boundary consists of one or more curve(s), or of a three-dimensional surface, again whose boundary consists of at least one curve. For a cubature problem, we want to determine the exact volume of a three-dimensional solid bounded at least in part by curved surfaces. Today, the use of the term quadrature hasn’t changed much: mathematicians, scientists, and engineers commonly say that they have “reduced a problem to a quadrature,” and mean that they have taken a complicated problem, simplified it by various means, and now the problem can be solve by evaluating an integral.

Historically, Hippocrates of Chios (ca. 440 B.C.) performed the first quadratures when he found the areas of certain lunes, regions that resemble the moon at about its first quarter. Antiphon (ca. 430 B.C.) claimed that he could "square the circle" (i.e. find the area of a circle) with an infinite sequence of inscribed regular polygons: first, a square; second, an octagon; next, a 16-gon; etc., etc. His problem was the "etc., etc.." Because Antiphon's quadrature of the circle required an infinite number of polygons, it could never be finished. He would have had to use the modern concept of the limit to produce a rigorous mathematical completion of this process. But Antiphon did have the start of a major idea, now called the method of exhaustion. More than 2000 years later, we credit Eudoxus (ca. 370 B.C.) with the development of the method of exhaustion: a technique of approximating the area of a region with an ever increasing number of polygons, with the approximations improving at each step and the exact area being attained after an infinite number of these steps; this technique has been modified to attack cubatures also.

Archimedes (287--212 B.C.), the greatest mathematician of antiquity, used the method of exhaustion to find the quadrature of the parabola. Archimedes approximated the area with a large number of ingeniously constructed triangles and then used a double reductio ad absurdum argument to prove the result rigorously and avoid any of the metaphysics of the infinite. For the circle, Archimedes first showed that the area depends upon the circumference; this is very easy for us to verify today, since both formulas depend on $p$ . Then, Archimedes approximated the area of the circle of unit radius using both inscribed and circumscribed regular 96-gons! His famous result was $3 \frac{10}{71} < p < 3 \frac{1}{7}$; but as these were only approximations, in the strict sense, they were not quadratures. This technique refined the method of exhaustion, so that when there are an infinite number of polygonal approximations, it is called the method of compression. Archimedes' process for finding the area of a segment of a spiral was to compress this region between sectors of inscribed and circumscribed circles: his method of determining the volume of a conoid (a solid formed by revolving a parabola around its axis) was to compress this solid between inscribed and circumscribed cylinders. In each case, the final step that rigorously established the result was a double reductio ad absurdum argument.

In perhaps his most famous work of all, a tract combining mathematics and physics, Archimedes employed indivisibles to estimate the center of gravity of certain two-dimensional regions and three-dimensional solids. (Archimedes acknowledged that while this work very strongly suggested the truth of his results, it also lacked full logical rigor.) If we consider one of these regions to be composed of an infinite number of straight lines, of varying lengths, then these lines are called indivisibles. Similarly, when the composition of a three-dimensional solid is thought of as an infinite number of circular disks, of varying radii but with zero thickness, then these disks are known as indivisibles.

Muslim mathematicians of the 9th through the 13th centuries were great students of Archimedes, but they never knew about Archimedes’ determination of the volume of a conoid. So, one of the most notable of all Arabic mathematicians, Thabit ibn Qurrah (826--901) devised his own rather complicated cubature of this solid, and then the Persian scientist Abu Sahl al-Kuhi (10th century) considerably simplified Thabit’s process. Ibn al-Haytham (965--1039), known in the West as Alhazen and famous for his work in optics, used the method of compression to find the volume of the solid formed by rotating the parabola around a line perpendicular to the axis of the curve.

During medieval times in the West, progress was made in applying the ideas of calculus to problems of motion. William Heytesbury (fl. 1335), a member of the notable group of scholars at Merton College, Oxford, first devised methods for the determination of the velocity and then the distance traveled of a body that was assumed to be in "uniform acceleration." Today, we can achieve these results by finding two indefinite integrals, or antiderivatives, in succession. News of this work of Heytesbury and his Merton colleagues reached Paris later in the 14th century where Nicole Oresme (1320–1382) represented both velocities and times as line segments of varying lengths. Oresme packed a body’s velocity lines together vertically, much like Archimedes’ indivisibles, over a horizontal base line, and the whole configuration, as he called it, represented the total distance covered by the body. In particular, the area of this
configuration was called the "total quantity of motion" of the body. Here we have precursors to modern graphs, and the birth of kinematics.

As Europeans began seriously to explore the globe, they wished to have a map of the world on which certain straight lines would represent the rhumb lines on the earth's surface. There have been several solutions to this problem, but the most famous solution was the Mercator projection, even though Gerard Mercator (1512--1594) did not explain its geometric principles. That task was taken up by Edward Wright (1561--1615) who, in addition, provided a table that showed that distances along rhumb lines would be closely approximated by summing the products \((\sec \varphi \Delta \varphi)\), where \(\varphi\) is the latitude; i.e., by approximating the integral of \(\sec \varphi\).

In his New Stereometry of Wine Barrels (1615), the famous astronomer Johannes Kepler (1571--1630) approximated the volumes of many three-dimensional solids, each of which was formed by revolving a two-dimensional region around an axis line. For each of these volumes of revolution, he subdivided the solid into many very thin slabs or disks called infinitesimals (note the difference between infinitesimals and Archimedes' indivisibles). Then, in each case, the sum of these infinitesimals approximated the desired volume. Kepler's Second Law of Planetary Motion required quadratures of segments of an ellipse, and to approximate these areas, he summed up infinitesimal triangles.

Bonaventura Cavalieri (1598--1647), a student of Galileo, developed a whole theory of indivisibles. For a two-dimensional region, Cavalieri considered the collection of "all the lines" to be a single number, the area of the region. Christiaan Huygens (1629--1695) criticized, "As to Cavalierian methods: one deceives oneself if one accepts their use as a demonstration but they are useful as a means of discovery preceding a demonstration ...that is what comes first...." Evangelista Torricelli (1608--1648), another disciple of Galileo and a friend of Cavalieri, attempted to reconcile some of the difficulties with indivisibles by asserting that lines could have some sort of thickness. He was careful to use reductio ad absurdum arguments to prove quadratures he obtained by indivisibles. "Gabriel's Horn" is an "incredible" cubature that Torricelli discovered.

Pierre Fermat (1601--1665) devised a technique for finding the areas under each of the "higher parabolas" \((y = kx^n)\), where \(k > 0\) is constant and \(n = 2, 3, 4, \ldots\) using narrow inscribed and circumscribed rectangles to lead to the method of compression. Then he employed a geometric series to do the same for each of the curves \(y = kx^n\), for \(n = -2, -3, -4, \ldots\). But, to his disappointment, he was never able to extend these processes to the "higher hyperbolas", \(y^m = kx^n\). By the 1640s, the general formula for the integral of the higher parabolas was known to Fermat, Blaise Pascal (1623--1662), Gilles Personne de Roberval (1602--1675), René Descartes (1596--1650), Torricelli, Marin Mersenne (1588--1648), and probably others.

John Wallis (1616--1703) was strongly committed to the relatively new algebraic notation, whose development was such a feature of 17th century mathematics. For instance, he treated the parabola, ellipse, and hyperbola as plane curves defined by equations in two variables rather than as sections of a cone. He also invented the symbol \(\int\) for infinity and, in using it, obscured places where we now know he should have used the limit. He extended the quadrature formula for \(y = kx^n\) to the cases when \(n\) was a positive rational number by using indivisibles, clever ratios, and appeals to reasoning by analogy. Wallis' dependence on formulas led him to a number of interesting quadratures.

Roberval exploited Cavalieri's Principle to find the area under one arc of the cycloid. Roberval and Pascal were the first to graph the sine and cosine functions and to find the quadratures of these curves (for the first quadrant). Pascal approximated double and triple integrals using triangular and pyramidal sums. But these were not cubatures, rather they were steps in his effort to calculate the moments of certain solids for each of which he then determined the center of gravity.

Finally, Gregory St. Vincent (1584--1667) determined the area under the hyperbola, \(xy = 1\), by using narrow inscribed and circumscribed rectangles of specially designed unequal widths and the method of compression. St. Vincent extended this and other quadratures to find many cubatures. Very shortly thereafter, his student, Alfonso Antonio de Sarasa (1618--1667) recognized that the quadrature of the hyperbola is closely connected to the product property of the logarithm.

Following a suggestion of Wallis, in 1657, William Neile (1637--1670) determined the length of an arbitrary section of the semi-cubical parabola, \(y^2 = x^3\), and in 1658, Christopher Wren (1632--1723), the famous architect, found the length of one arch of the cycloid. In 1659, Hendrick van Heuraet (1634--ca.1660) generalized this work by summing infinitesimal tangents to a curve, thereby deriving the essence of our modern method of rectification – using an integral to find the length of an arc.

In geometric form, much of the calculus of the first two-thirds of the 17th century culminated in The
To Gottfried Wilhelm Leibniz (1646--1716), a curve was a polygon with an infinite number of sides. Leibniz (1686) let \( y \) represent an ordinate of the curve and \( dx \) the infinitesimal distance from one ordinate to the next, i.e., the difference between "successive" abscissae. Then he said, "I represent the area of a figure by the sum of all the \([\text{infinitesimal}]\) rectangles contained by the ordinates and the differences of the abscissae ... and thus I represent in my calculus the area of the figure by \( \int y \, dx \)." Leibniz took the elongated "S" for the integral from the Latin \textit{summa} and the \( d \) from the Latin \textit{differentia}, and these have remained our most basic calculus notations ever since. He considered calculus computations to be a means of somehow abbreviating the classical Greek method of exhaustion. Leibniz was ambivalent about infinitesimals, but he believed that formal calculus computations could be trusted because they yielded correct results.

The term \textit{integral}, as we use it in calculus, was coined by Johann Bernoulli (1667--1748) and first published by his elder brother Jakob Bernoulli (1654--1705). Mainly as a consequence of the power of Newton's and Leibniz's Fundamental Theorem of Calculus, integrals were simply regarded as "reverse" \textit{derivatives}. Area was an intuitive notion, quadratures that could not be found using the Fundamental Theorem of Calculus were approximated. Even though Newton had made a very imperfect stab at the idea of a limit, no one in the 17th or 18th centuries had the foresight to combine \textit{limits} and areas to define the integral mathematically. Instead, with great ingenuity, many clever integration formulas were developed. At about the same time as Newton's table of integrals was published, Johann Bernoulli devised systematic procedures for integrating all rational functions, what we now call the method of \textit{partial fractions}. These rules were neatly summarized in Leonhard Euler's (1707--1783) encyclopedic three-volume work on integral calculus (1768--1770). Incidentally, these efforts stimulated increased interest during the 18th century in factoring and solving higher degree polynomial equations.

While describing the paths of comets in the \textit{Principia Mathematica} (1687), Newton posed a problem with major implications for calculus: "To find a curved line of the parabolic kind [i.e., a polynomial] which shall pass through any given number of points." Newton rediscovered the interpolation formula of James Gregory (1638--1675); today, it is called the Gregory-Newton formula, and in 1711, he pointed out its importance: "Hence the areas of all curves may be nearly found ... the area of the parabola [polynomial] will be nearly the same with the area of the curvilinear figure ... the parabola [polynomial] can always be squared geometrically by methods generally known [i.e., by using the Fundamental Theorem of Calculus]." Newton's interpolation work was extended at various times by Roger Cotes (1682--1716), James Stirling (1692--1770), Colin Maclaurin (1698--1746), Leonhard Euler, and others. In 1743, the self-educated mathematician Thomas Simpson (1710-1761) found what has become a popular and useful special case of the Newton-Cotes formula for approximating an integral, Simpson's Rule.

Though Euler had made calculus more analytic than geometric with his emphasis on functions (1748; 1755; 1768), there were many misunderstandings about the function concept itself in the 18th century. Certain physics problems, such as the \textit{vibrating string problem}, contributed to this confusion. Euler so closely identified functions with analytic expressions that he thought of a continuous function as being defined by only one formula for its whole domain. The modern idea of a \textit{continuous function}, independent of any formula(s), was initiated in 1791 by Louis-François Arbogast (1759--1803): "The law of continuity consists in that a quantity cannot pass from one state [value] to another [value] without passing through all the intermediate states [values] ..." This insight was made rigorous in an 1817 pamphlet by Bernhard Bolzano (1781--1848) and is now know as the \textit{Intermediate Value Theorem}. Discontinuous functions (in the modern sense) were forced onto the mathematical and scientific community by Joseph Fourier (1768--1830) in his famous \textit{Analytical Theory of Heat} (1822).

When Augustin Louis Cauchy (1789--1857) undertook the total reform of calculus for his engineering students at the École polytechnique in the 1820s, the integral was one of his cornerstones:

In the integral calculus, it has appeared to me necessary to demonstrate generally the existence of the \textit{integrals or primitive functions} before making known their diverse
properties. In order to attain this object, it was found necessary to establish at the outset
the notion of integrals taken between given limits or definite integrals.

Cauchy defined the integral of any continuous function on the interval \([a,b]\) to be the limit of the sums of
areas of thin rectangles. His first obligation was to prove that this limit existed for all functions continuous
on the given interval. Unfortunately, though Cauchy made use of the Intermediate Value Theorem, he
failed in this task because he overlooked two subtle but crucial theoretical facts. He was not aware of
these logical gaps in his reasoning, and he went on to justify the Mean Value Theorem for Integrals and to
prove the Fundamental Theorem of Calculus for continuous functions. Niels Henrik Abel (1802--1829) also
pointed out certain delicate errors when using Cauchy’s integral to integrate every term of an infinite
series of functions.

The first rigorous proof of the convergence of general Fourier series was devised by Peter Gustav Lejeune
Dirichlet (1805--1859) in 1829. Dirichlet is also responsible for the modern definition of function (1837).
In 1855, Dirichlet succeeded Carl Friedrich Gauss (1777-1855) as professor at the University of Göttingen.
In turn, Georg F. B. Riemann (1826--1866) succeeded Dirichlet’s (1859) at Göttingen. In the process of
extending Dirichlet’s work on Fourier series, Riemann generalized Cauchy’s definition of the integral to
arbitrary functions on the interval \([a,b]\), and the limit of Riemann sums is the formulation in the text.
Immediately, Riemann asked, “in what cases is a function integrable?” Most of Cauchy’s development of
the theory of integration was subsequently verified by Riemann and others, but there were still difficulties
with integrals and infinite series that were not worked out until the early years of the 20th century.
The Fundamental Theorem of Calculus

When we think of the geometric origins of derivatives and integrals, tangent lines to curves and areas, respectively, there is no clue that would suggest the Fundamental Theorem of Calculus. To Eudoxus (ca. 370 B.C.), Euclid (ca. 300 B.C.), Archimedes (287--212 B.C.), Apollonius (ca. 262--190 B.C.), and all of the other mathematicians of classical antiquity, tangent lines, quadratures, and cubatures had no special status among their other geometric problems and theorems. To these mathematicians, the Fundamental Theorem of Calculus probably would have been a great surprise.

When algebra was first used to describe curves in the analytic geometry of René Descartes (1596--1650) and Pierre Fermat (1601--1665), we can see the first faint glimmers of a connection between tangents and quadratures. In his study of the "higher parabolas," \( y = kx^n \), where \( k \) is constant and \( n = 2, 3, 4, \ldots \), Fermat derived the formula \( y/k \), for the subtangent at any point on the curve. From this, and from our standpoint today, it would have been easy to find the formula for the derivative; but to Fermat, \( nx^{n-1} \) was not the goal. Some time in the 1640s, Fermat showed that the area between any one of the higher parabolas and the horizontal axis, for \( 0 \leq x \leq a \), was equal to the area of the rectangle of width \( a \) and height \( a^n/(k+1) \). Today, we can see that Fermat was tantalizingly close to the Fundamental Theorem of Calculus, as it would have been expressed in terms of his higher parabolas. But that did not appear to be of any interest to him.

Fermat also anticipated the Fundamental Theorem of Calculus in the procedure he devised to find the center of gravity of a conoid (now known as a paraboloid of revolution), a problem that had originated with Archimedes. But, he failed to recognize this. To find the center of gravity of a more general solid, Gilles Personne de Roberval (1602--1675) used a summation process and both the tangent and the quadrature of certain curves, but he also missed the connection. Gregory St. Vincent (1584--1667) and Evangelista Torricelli (1608--1647) added to Roberval's technique for determining centers of gravity without perceiving any further important mathematical principles. In notes left unpublished at his death, Torricelli derived his construction of the tangent lines to the "higher hyperbolas" of Fermat, \( y^m = kx^n \), from the quadrature of these curves, but without any hint of more general and larger ideas. Additionally, Torricelli related the construction of tangent lines to spirals, \( r^m = k \theta^n \), in polar coordinates, to the quadrature of the spiral. Torricelli's results were well known to his students, most prominently Vincenzo Viviani (1622--1703), and through them to James Gregory (1638--1675) and Isaac Barrow (1630--1677) when the latter traveled and studied in Italy; in this way, many of the techniques of what we now call calculus were transmitted to England.

In a roundabout way involving rectification (finding the length of a segment of a curve), James Gregory considered the area between the curve, \( y \), and the \( t \)-axis, beginning at \( t = a \) as a function of the right endpoint, \( t = x \). Then he found the tangent line to this new curve at \( t = x \) and showed that its slope at this point was equal to the ordinate, \( y \), of the original curve. This convoluted process brought Gregory within sight of Part 1 of the Fundamental Theorem of Calculus as it is stated in Thomas' Calculus. But this result was just a small part of Gregory's Universal Part of Geometry (1668), his attempt to summarize and organize the geometry of the calculus as he knew it (and much of which he had learned during his studies in Italy, 1664--1668). Not only was he not looking for the Fundamental Theorem of Calculus here, he was not even using the convenient formulas we use today.

Isaac Barrow was the first Lucasian Professor of Mathematics and Natural Philosophy at Cambridge (1663--1669). Due to the similarities in the educations and backgrounds of their writers, Barrow's Geometrical Lectures (1670) and Gregory's Universal Part of Geometry covered much of the same ground. Barrow's work dug somewhat deeper into the 17th century efforts leading to the development of calculus. In particular, in several places, Barrow displayed at least an intuitive grasp of the fact that tangents and quadratures were inverse operations. In discussing velocity and distance, he showed how the tangent line to one curve (distance) could lead to the construction and quadrature of another curve (velocity), and conversely. In his most comprehensive treatment, Barrow first showed geometrically that the area between an increasing but otherwise arbitrary curve, \( f(t) \), and the horizontal axis, \( a \leq t \leq b \), was equal to \( y \) times the subtangent of an auxiliary curve, \( h(x) \), where \( y \) is the ordinate of the given curve at \( t = x \). Barrow's geometrical language probably concealed the fact that his \( h(x) \) was actually a constant multiple of what we now call the antiderivative, \( F(x) \). In this way, he had also anticipated Part 1 of the Fundamental Theorem of Calculus. Further on in his Geometrical Lectures, Barrow proved a theorem relating the sum of the infinitesimal rectangles comprising the region between a curve and the horizontal axis, \( a \leq x \leq b \), to the rectangle whose width is a constant and whose height is \( F(b) - F(a) \) in modern notation. This is the essence of Part 2 of the Fundamental Theorem of Calculus as found in Thomas'
Calculus. Barrow’s Geometrical Lectures was the culmination of the 17th century geometrical processes that led to our modern derivative and integral. Though his student and protégé, Isaac Newton (1642--1727), encouraged him to include some additional algebraic methods in his work, Barrow was a geometer at heart and a very talented one. And so he fell just short of the profound insight that the calculus, through the Fundamental Theorem, is a single intellectual entity.

Thanks to the foundations provided by Barrow, Isaac Newton (1642--1727) mastered the tangent and quadrature results of the first two-thirds of the 17th century. In a letter to Gottfried Wilhelm Leibniz (1646--1716), Newton clearly stated, in terms of physics, what the two most basic problems of calculus were (and still are): "1. Given the length of the space continuously [i.e., at every instant of time], to find the speed of motion [i.e., the derivative] at any time proposed. 2. Given the speed of motion continuously, to find the length of the space [i.e., the integral or the antiderivative] described at any time proposed." But in place of derivatives, Newton employed fluxions of variables, denoted for instance by \(x\), and instead of antiderivatives, he used what he called fluents.

From Gregory, Newton took the idea that the area between a curve, \(y\), and the horizontal axis was dependent upon the right endpoint, \(t = x\). In fact, Newton thought of the area as being actually generated by the movement of the vertical line \(t = x\). So the fluxion of the area was simply \(yx\). Thus, Newton’s technique for finding such quadratures was to find the fluent of \(y\), equivalent to finding our antiderivative; this is the core of Part 2 of the Fundamental Theorem of Calculus as found in Thomas’ Calculus. Newton used the Fundamental Theorem to find exact values for many areas, just as we do today. In general, Newton began thinking of the geometric problems of calculus in algebraic terms.

Newton synthesized nearly all of the earlier work on calculus. In addition to Barrow, he was especially influenced by René Descartes’ (1596--1650) La géométrie in the Latin translation and with the additions by Frans van Schooten (1615-1660), and by John Wallis’ (1616--1703) The Arithmetic of Infinites. He crowned this study with his own genius, writing three book-like manuscripts on calculus: the first was written in October 1666; the second in 1669; and the third, updating his previous two, in 1671. In his most famous work, the Principia Mathematica (1687), Newton used the ideas and some of the techniques of calculus, but because the Principia was written in mostly geometric form, the formulas and algebraic parts of calculus were absent. Newton’s three calculus monographs, however, were widely known through copies made for his colleagues of the Royal Society. But they were not published until much later, after his death.

When Leibniz came to Paris in 1672 on a diplomatic mission, he was introduced to the emerging ideas of calculus by Christiaan Huygens (1629--1695), a member of the new French Academy. Leibniz studied several of the works of authors on higher mathematics, and he reported that those of Blaise Pascal (1623--1662) were especially helpful. Most of Leibniz’s writings on calculus fall into three groups: his manuscripts – almost diaries – begun while he was in Paris (1672--1676); the articles he published in the journal Acta Eruditorum in the 1680s and 1690s; and a manuscript, History and Origin of the Differential Calculus (1714), about which we shall have more to say in a moment.

Leibniz’s ideas about integrals, derivatives, and calculus in general were derived from close analogies with finite sums and differences. For instance, for the Fundamental Theorem of Calculus, if he were given a finite sequence of numbers such as, \(Y\): 0, 1, 8, 27, 64, 125, and 216, with differences \(y\): 1, 7, 19, 37, 61, and 89, he noticed that the sum of the differences, \(a y = (1 - 0) + (8 - 1) + (27 - 8) + ... + (216 - 125)\) telescoped into the difference between the first and the last values of \(Y\), 216 -- 0. Now, to Leibniz, a curve was a polygon made up of an infinite number of sides, each "infinitesimally" long. Thus, he wrote in 1680, "I represent the area of a figure by the [infinite] sum of all the rectangles contained by the ordinates and the differences of the abscessae," i.e., as \(\int ydx\). Then, "mounting to greater heights" by drawing on the analogy with finite sums and differences, Leibniz claimed that when finding the area represented by \(\int ydx\) one should find a curve \(Y\) such that the ordinates \(y\) are differences of \(Y\), or \(y = dY\).

In modern terms, \(Y\) is our antiderivative, and so Leibniz has formulated an early statement of Part 1 of the Fundamental Theorem of Calculus in Thomas’ Calculus. Later, in a 1693 paper in the Acta Eruditorum, Leibniz wrote, "the general problem of quadratures can be reduced to the finding of a curve that has a given law of tangency," and he went on to specify this law in the form of Part 2 of the Fundamental Theorem of Calculus.

To fully appreciate Leibniz’s contributions to calculus, one must consider their context within his significant work in logic, metaphysics, and philosophy because he regarded all of these activities as interrelated. For Leibniz, the existence of infinitesimals might have been a worthwhile philosophical problem, but it missed the point of his calculus. Calculus, and especially the Fundamental Theorem, "contained a handy means of reckoning" and was an abbreviation of the rigorous methods of tangents and quadratures of Archimedes (287--212 B.C.) and other classical Greek geometers. On the other hand, Jakob (1654--1705) and Johann Bernoulli (1667--1748) and the other 18th century mathematicians and scientists who took advantage of
Leibniz’s calculus, especially its fertile notation, freely used, expanded, and applied the calculus, often with spectacular results.

Very unfortunately, about the turn of the 18th century, a few of Newton’s followers attacked Leibniz, accusing him of plagiarizing his calculus from Newton during his London visits of 1673 and 1676. Newton and Leibniz never met face-to-face, but during the first decades of the 18th century, Newton was president of the Royal Society and Leibniz was still a member. Leibniz composed his "History and Origin of the Differential Calculus" (1714) for his defense, but to no avail. The matter turned into a priority dispute on a titanic scale, and it became an ugly discredit to all the participants as the 18th century wore on; for instance, in misguided loyalty, most English mathematicians confined themselves to Newton’s fluxions and fluents, and avoided Leibniz’s superior notations until the early years of the 19th century. The consensus today, after much close and fair study by many scholars, is that Newton and Leibniz devised the Fundamental Theorem of Calculus independently, and that therefore they should share equally in the glory of the creation of calculus.

Leibniz had argued for the Fundamental Theorem of Calculus by analogy, and Newton had based his justification for it upon fluxions and fluents, which in turn depended on the intuition of points moving along a curve. Colin Maclaurin (1698--1746) proved Part 1 of the Fundamental Theorem for simple power functions, \( y = x^n \), where \( n = 1, 2, 3, \ldots \), and Joseph Louis Lagrange (1736--1813) extended Maclaurin’s basic idea to increasing functions represented by a power series. The modern proof of the Fundamental Theorem of Calculus was formulated for functions continuous on \( a \leq x \leq b \) by Augustin Louis Cauchy (1789-1857) in his *Lessons Given at the École Royale Polytechnique on the Infinitesimal Calculus* (1823). The arguments that Cauchy gave are the same as those found in *Thomas’ Calculus*. With his Fundamental Theorem of Calculus, Cauchy supplied the keystone, for all continuous functions, which finally rigorously united the two major branches of calculus into one structure, both elegant and useful.
History of Differential Equations

In many ways, differential equations are the heart of analysis and calculus, two of the most important branches of mathematics for over the past 300 years. Differential equations are an integral part or a goal of many undergraduate calculus courses. As an important mathematical tool for the physical sciences, the differential equation has no equal. So it is widely accepted that differential equations are important in both applied and pure mathematics. The history of this subject is rich in its development, and that’s what we look at here.

The foundations of this subject seem to be so dominated by the contributions of one man, Leonhard Euler, that one could say the history of this subject starts and ends with him. Of course, that would be a gross simplification of its development. There are many important contributors, and those who came before Euler were necessary so that Euler could understand the calculus and analysis necessary to develop many of the fundamental ideas. The contributors after Euler have both refined his work and forged entirely new ideas, inaccessible to Euler’s 18th-century perspective and sophisticated beyond the understanding of just one person.

This is the story of the development of the subject of differential equations. We take a brief look at the people, the equations, the techniques, the theory, and the applications.

The story begins with the developers of the calculus, Fermat, Newton, and Leibniz. As soon as these brilliant mathematicians had sufficient understanding and notation for the derivative, the derivative was soon appearing in equations and the subject was born. However, they soon found that solutions for these equations weren't so easy. The symbolic manipulations and simplifications of algebra helped only so much. The integral (antiderivative) and its theoretical role in the Fundamental Theorem of Calculus offered direct help only when the variables were separated in very special circumstances. The separation of variables method was developed by Jakob Bernoulli and generalized by Leibniz. So these early developers in the 17th century focused on these special cases and left a more general development of theories and techniques for those who followed.

Around the beginning of the 18th century, the next wave of differential equations developers began to apply these kinds of equations to problems in astronomy and the physical sciences. Jakob Bernoulli carefully studied and wrote differential equations for planetary motion, using the principles of gravity and momentum developed by Newton. Bernoulli's work included the development of the catenary and the use of polar coordinates. By this time, differential equations were interfacing with other kinds of mathematics and science to solve meaningful applied problems. Halley used the same principles to analyze the motion of a comet that today bears his name. Jakob's brother, Johann Bernoulli, was probably the first mathematician to understand Leibniz's calculus and the principles of mechanics to mathematically model physical phenomena using differential equations and find their solutions. Ricatti (1676--1754) began serious study of a particular equation, but he found himself confined by the theories of the time to only special cases of the equation that today bears his name. The Bernoullis, Jakob, Johann, and Daniel, all studied the cases of the Ricatti equation as well. By the time Taylor used series to "solve" differential equations, others had developed and used these series for various purposes. However, Taylor's development of finite differences began a new branch of mathematics closely related to the development of differential equations. By the early 18th century, these and many other mathematicians had assembled a growing array of techniques to analyze and solve several varieties of differential equations. However, most equations still were unknown in terms of properties or solution methods. Fifty years of differential equations had brought considerable progress, but no general theory.

The development of the subject of differential equations needed a master to consolidate and generalize the existing methods and create new, more powerful techniques to tackle large families of equations. Many equations looked approachable but turned out to be deceptively difficult. In many cases, solution techniques had evaded pursuers for nearly 50 years, when Leonhard Euler arrived on the differential equations scene. Euler had the benefit of the earlier work, but the key to his understanding was his knowledge and perception of functions. Euler understood the role and structure of functions, and he studied their properties and definitions. He soon found that functions were the key to understanding differential equations and developing methods for their solutions. Using his knowledge of functions, he developed general procedures for solutions of many kinds of equations. He was the first to understand the properties and roles of the exponential, logarithmic, trigonometric, and many other elementary functions. Euler also developed many new functions based on series solutions of special types of differential equations. His techniques of conjecturing and finding undetermined coefficients were major
steps in the development of this subject. In 1739, he developed the method of variation of parameters. His work also included the use of numerical approximations and the development of numerical methods, which provided approximate "solutions" for nearly all equations. Euler then continued on to perform applied work in mechanics that led to differential equation models and solutions. He was the master that this subject needed to develop past its primitive beginnings to a cohesive subject central to the development of modern applied mathematics.

After Euler came many specialists that refined or extended many of Euler's ideas. In 1728, Daniel Bernoulli used Euler's methods to help him study oscillations and the differential equations that produce these kinds of solutions. D'Alembert's work in mathematical physics involved partial differential equations and exploration for solutions of the most elementary forms of these equations. Lagrange followed closely in Euler's footsteps, developing more theory and extending results in mechanics, especially equations of motion (3-body problem) and potential energy. Lagrange's greatest contributions were probably in function definition and properties, which kept the momentum going for generalizing methods and analyzing new families of differential equations. Lagrange was probably the first mathematician with enough theory and tools to be a true analyst of differential equations. In 1788, he introduced general equations of motion for dynamical systems, today known as Lagrange's equations. Laplace's work on the stability of the solar system led to more advancements, including better numerical techniques and a better understanding of integration. In 1799, he introduced the ideas of a Laplacian of a function. Laplace clearly recognized the roots of his work when he wrote "Read Euler, read Euler, he is the master of us all." Legendre's work on differential equations was motivated by projectile motion, for the first time taking into account new factors such as air resistance and initial velocities. Lacroix was next to make his mark. He worked on advances in partial differential equations and incorporated many of the advances since Euler's time in his textbooks. Lacroix's major contribution was in synthesizing many of the results of Euler, Lagrange, Laplace, and Legendre. Next in line was Fourier. His mathematical research made contributions in the study and computation of heat diffusion and the solution of differential equations. Much of that work appears in Fourier's *The Analytical Theory of Heat* (1822), in which he made extensive use of the series that bear his name. This result was an important tool for the study of oscillations.

Fourier, however, contributed little to the mathematical theory of these series, which were well known much earlier to Euler, Daniel Bernoulli, and Lagrange. Charles Babbage's contributions came through a different route. He developed a calculating machine called the Difference Engine that used finite differences to approximate solutions to equations.

The next major breakthrough for this subject occurred in the early 19th century, when the theories and concepts of functions of complex variables were developed. The two major contributors to this development were Gauss and Cauchy. Gauss used differential equations to improve the theories of planetary orbits and gravitation. Gauss established potential theory as a coherent branch of mathematics. He also recognized that theory of functions of a complex variable was the key to understanding many of the results needed in applied differential equations. Cauchy applied differential equations to model the propagation of waves on the surface of a liquid; the results are now classics in hydrodynamics. He invented the method of characteristics, which is important in the analysis and solution of many partial differential equations. Cauchy was the first to define fully the ideas of convergence and absolute convergence of infinite series, and he initiated the rigorous analysis of calculus and differential equations. He was also the first to develop a systemic theory for complex numbers and to develop the Fourier transform to provide for algebraic solutions to differential equations.

After these major contributions of Gauss and Cauchy, others were able to refine these powerful theories and apply them to various branches of science. Poisson's early work in mechanics appeared in *Traité de mécanique* in 1811. He applied his understanding of differential equations to applications in physics and mechanics, including elasticity and vibrations. Much of his original work was done in the solution and analysis of differential equations. Another applier of these theories was George Green. Green's work on the mathematical foundations of gravitation, electricity, and magnetism was published in 1828 in *An Essay on the Application of Mathematical Analysis to Electricity and Magnetism*. Green's mathematics provided the foundation on which Thomson, Stokes, Rayleigh, Maxwell, and others built the present-day theory of electromagnetism. Bessel was a friend of Gauss and applied his knowledge in differential equations to astronomy. His work on the Bessel functions was done to analyze planetary perturbations. Later these constructs were used to solve differential equations. Ostrogradsky collaborated with Laplace, Legendre, Fourier, Poisson, and Cauchy while using differential equations to develop theories on the conduction of heat. Joseph Liouville was the first to solve boundary value problems by solving equivalent integral equations, a method refined by Fredholm and Hilbert in the early 1900s. Liouville's work on the theory of integrals of elementary functions was a substantial contribution to solutions of differential equations. Stokes' theoretical and experimental investigations covered hydrodynamics, elasticity, light, gravity, sound, heat, meteorology, and solar physics. He used differential equation models in all these fields of study.
By the middle of the 19th century, a new structure was needed to attack systems of more than one differential equation. Several mathematicians came to the rescue. **Jacobi** developed the theory of determinants and transformations into a powerful tool for evaluating multiple integrals and solving differential equations. The structure of the Jacobian was developed in 1841. Like Euler, Jacobi was a remarkably skilled calculator and an expert in a variety of applied fields. **Cayley** also worked on determinants and created a theory for matrix operations in 1854. Cayley was a friend of J. J. Sylvester and came to America to lecture at Johns Hopkins University in 1881--1882. Cayley published over 900 papers covering many areas of mathematics, theoretical dynamics, and astronomy. Cayley originated the notion of matrices in 1858 and did a great deal of development of matrix theory over the next several decades. **Josiah Gibbs** made contributions to thermodynamics, electromagnetics, and mechanics. For his foundational work in systems of equations, Gibbs is known as the father of **vector analysis**.

As the end of the 19th century approached, the major efforts in differential equations moved into a more theoretical realm. In 1876, Lipschitz (1832--1903) developed existence theorems for solutions of first order differential equations. **Hermite’s** work was in developing theory of functions and solutions of equations. As the theory developed, the six basic trigonometric functions were shown to be transcendental, as were the inverses of the trigonometric functions and the exponential and logarithmic functions. Hermite showed that the quintic equation could be solved by elliptic functions. While his work was theoretical, Hermite polynomials and Hermite functions were later shown to be very useful in solving Schrödinger’s wave equation and other differential equations. Next to build theoretical foundation was **Bernhard Riemann**. His doctorate was obtained under the direction of Gauss in the theory of complex variables. Riemann also had the benefit of working with physicist Wilhelm Weber. Riemann’s work in differential equations contributed to results in dynamics and physics. In the late 1890s, Gibbs wrote an article that described the convergence and the ”Gibb's phenomenon” of Fourier series. The next major theoretical contributor was **Kovalevsky**, the greatest woman mathematician prior to the twentieth century. After overcoming considerable difficulties because of discrimination over her gender, she had the opportunity to study with Weierstrass. Early in her research, she completed three papers on partial differential equations. In her study of the form of Saturn's rings, she built on the work of Laplace, whose work she generalized. Kovalevsky primarily worked on the theory of partial differential equations, and a central result on the existence of solutions still bears her name. She published numerous papers on partial differential equations. Later in the 20th century, theoretical work by Fredholm and Hilbert refined the earlier results and developed new classifications for further understanding some of the most complicated families of differential equations.

The next major thrust was in developing more robust and efficient numerical methods. **Carl Runge** developed numerical methods for solving the differential equations that arose in his study of atomic spectra. These numerical methods are still used today. He used so much mathematics in his research that physicists thought he was a mathematician, and he did so much physics that mathematicians thought he was a physicist. Today his name is associated with the Runge-Kutta methods to numerically solve differential equations. Kutta, another German applied mathematician, is also remembered for his contribution to the differential equations-based Kutta-Joukowski theory of airfoil lift in aerodynamics. In the latter half of the 20th century, many mathematicians and computer scientists implemented numerical methods for differential equations on computers to provide fast, efficient solutions for large-scale, complicated systems on complex geometries. **Richard Courant** and Garrett Birkhoff were successful pioneers of this effort.

Nonlinear equations were the next major hurdle. **Poincaré**, the greatest mathematician of his generation, produced more than 30 technical books on mathematical physics and celestial mechanics. Most of these works involved use and analysis of differential equations. In celestial mechanics, working from the results of American astronomer George Hill, he conquered the stability of orbits and initiated the qualitative theory of nonlinear differential equations. Many results of his work were the seeds of new ways of thinking, which have flourished, such as analysis of divergent series and nonlinear differential equations. Poincaré was able to understand and contribute in the four major areas of mathematics -- analysis, algebra, geometry, and number theory. He had creative command of the whole of mathematics as it existed in his day, and he was probably the last person ever to be in this position. In the 20th century, **George Birkhoff** used Poincaré’s ideas to analyze large systems of dynamical systems and establish a theory for the analysis of solution properties for these equations. By the 1980s, the emerging theory of chaos used the principles developed by Poincaré and his successors.
History of Sequences and Series

Zeno of Elea (490--425 B.C.) wrote a book of 40 paradoxes concerning the continuum and the infinite. At least four of the paradoxes have influenced the development of mathematics to explain the relevant phenomena. Unfortunately, the book has not survived to modern times, so we know of these paradoxes only through other sources. Zeno's paradoxes about motion baffled mathematicians for centuries. They ultimately involved the summing of an infinite number of positive terms to a finite number, which is the essence of convergence of an infinite series of numbers. Numerous mathematicians have contributed to the understanding of the properties of sequences and series. This essay outlines the contributions of some of those mathematicians who studied sequences and series.

Zeno wasn't the only ancient mathematician to work on sequences. Several of the ancient Greek mathematicians used their method of exhaustion (a sequential argument) to measure areas of shapes and regions. By using his refined reasoning technique called the "method," Archimedes (287--212 B.C.) achieved a number of remarkable results involving areas and volumes of various shapes and solids. He actually constructed several examples and tried to explain how infinite sums could have finite results. Among his many results was that the area under a parabolic arc is always two-thirds the base times the height. His work was not so complete or rigorous as the later developers of the mathematics of sequences and series, like Newton and Leibniz, but it was just as amazing. Although Archimedes was hampered by lack of precise and efficient notation, he was able to discover many of the elements of modern analysis of sequences and series.

The next major contributor to this area of mathematics was Fibonacci (1170--1240). He discovered a sequence of integers in which each number is equal to the sum of the preceding two, introducing it in terms of modeling a breeding population of rabbits. This sequence has many curious and remarkable properties and continues to find application in many areas of modern mathematics and science. During this same period, Chinese astronomers developed numerical techniques to analyze their observation data. During the 13th and 14th centuries, Chinese mathematicians used the idea of finite differences to help analyze trends in their data. Today, methods like theirs are used to understand long-term behavior and limits of infinite sequences. This early work in Asia led to further investigation and analysis of numerous progressions and series but had little influence on European mathematicians.

Oresme (1325--1382) studied rates of change, like velocity and acceleration, using a sequential approach. His major work, De configurationibus, was the first to present graphs of velocities. The argument we use to show the divergence of the harmonic series was devised by Oresme in his publication. Two hundred years later, Stevin (1548--1620) advanced mathematics by providing a more understandable symbology. He understood the physical and mathematical conceptions of acceleration due to gravity. He summed series and analyzed sequences but stopped short of defining or explaining limits and convergence. Stevin's contemporary, Galileo (1564--1642) applied mathematics to the sciences, especially astronomy. Based on his study of Archimedes, Galileo improved the understanding of hydrostatics. He developed the results for free fall motion under gravity and the motion of the planets. He went so far as to suggest there may be a third property between the finite and infinite. Galileo left his successors with advice and challenges found in the following two quotations:

Where the senses fail us, reason must step in.

Infinities and indivisibles transcend our finite understanding, the former on account of their magnitude, the latter because of their smallness; imagine what they are when combined.

As the development of calculus was taking place, progress in the understanding of infinite series played a role in the development of both the differential and integral calculus. Pascal (1623--1662) was fascinated by the remarkable results that came from infinite sums but was bewildered by this concept. To him, the infinite was something to admire but impossible to understand. Pascal preferred St. Vincent's (1584--1667) geometric approach to series and their convergence over the new analytic approach of Fermat (1601--1665) and Descartes (1596--1650) that he could not visualize or understand. Despite Pascal's limitation in understanding series, he, along with Descartes and Fermat, used series calculations in their contributions to the foundations of differential and integral calculus.

Up until the middle of the 17th century, mathematicians had developed and analyzed series of numbers. The time had come to investigate sequences and series of functions. Both Newton (1642--1727) and Leibniz (1646--1716) developed several series representations for functions. Using algebraic and geometric methods, Newton calculated the series for the trigonometric functions sin(x) and cos(x) and for the exponential function. These results are found in Newton's works entitled Method of Fluxions and
**Infinite Series** and **Analysis with Infinite Series**. Newton used series in developing many calculus results, such as area, arc length, and volumes. Leibniz summed sequences of reciprocal polygonal numbers and, following the work of St. Vincent, summed and analyzed numerous geometric sequences. Leibniz used a sequential approach of infinitely close values to explain the limiting concept. Although he never thought of the derivative as a limit, he discovered many of the results we now study in calculus using limits.

**Brook Taylor** (1685--1731) was not the first to invent the structure and process we call Taylor series, and Maclaurin series were not developed by Colin Maclaurin (1698--1746). James Gregory (1698--1746) was working with Taylor series when Taylor was only a few years old. Gregory also published the Maclaurin series for many trigonometric functions before Maclaurin was born. Taylor was not aware of Gregory’s work when he published his book *Methodus incrementorum directa et inversa*, which contained what we now call Taylor series. Taylor had independently developed a calculus-based method to generate series representations of functions. Later, Maclaurin quoted Taylor's work in a calculus book he wrote in 1742. Maclaurin’s book popularized series representations of functions, and although Maclaurin never claimed to have discovered them, Taylor series centered at \( a = 0 \) later became known as Maclaurin series. **Johann Bernoulli** (1667--1748) also made an independent discovery of Taylor's theorem.

**Euler** (1707--1783) often used infinite series in his work to develop new methods or to model application problems. He published *Mechanica* in 1736, where he systematically applied calculus to mechanics and developed new methods to solve differential equations using power series. He established the summation notation we use today, using sigma for the summation symbol. **D'Alembert** (1717--1783) wrote five papers dealing with methods of integrating differential equations. Although he had received little formal scientific education, it is clear that he had become familiar with the works of Newton, L'Hospital, and the Bernoullis. D'Alembert published many works on mathematics and mathematical physics, culminating in his major work, *Traité de dynamique*. He considered the derivative as a limit of difference quotients, which put him ahead of his peers in understanding calculus. He also developed the ratio test to determine the convergence of many series. Through the work of d'Alembert, the nature of the research in series was slowly changing from practical calculations to a more theoretical foundation.

**Lagrange** (1736--1813) extended the work of Euler in the equations of motion and the understanding of potential energy. He published *Mécanique analytique* (1787), which applied calculus to the motion of objects. Lagrange's greatest work was in the theory and application of the calculus. He felt that Taylor series played the fundamental role in understanding calculus, although he still avoided the limit and convergence properties of sequences and series. **Bolzano** (1781--1848) confronted this issue, pointing out that convergence was important to understand and use series. He tried to explain convergence by associating it with the idea of bounded subsets. Bolzano was a believer in Lagrange's method of using Taylor series as the basis for calculus. **Fourier** (1768--1830) made contributions in the study and computation of heat diffusion and the solution of differential equations. *Théorie analytique de la chaleur* (*The Analytical Theory of Heat*, 1822) contains extensive use of the series comprised on trigonometric functions that today are called Fourier series. However, he contributed very little to the mathematical theory of these series, which were well known much earlier to Euler, Daniel Bernoulli, and Lagrange.

Finally, the mathematics community was motivated to establish a more theoretical foundation for the ideas of limit and convergence of sequences and series. **Cauchy** (1789-1857) was the first to define fully the ideas of convergence and absolute convergence of infinite series. This work was done in conjunction with the development of a rigorous analysis of calculus. He was also the first to develop a systematic theory for complex numbers and to develop the Fourier transform for differential equations. However, both Cauchy and his colleague **Niels Henrik Abel** (1802--1829) ignored the utility of divergent series. Abel wrote in 1828 "Divergent series are the invention of the devil, and it is shameful to base on them any demonstration whatsoever."

**Runge** (1856--1927) developed the sequence-based solution method to numerically solve differential equations along with M. W. Kutta (1867--1944). Sequences and series became standard tools for approximating functions and calculating results in numerical computing.

Self-educated Indian mathematician Srinivasa Ramanujan (1887--1920) used sequences and power series to develop results in number theory. Ramanujan’s work was theoretic and produced numerous important results used by mathematicians in the 20th century. His British collaborators Godfrey Harold (G.H.) Hardy (1877--1947) and John Littlewood (1885--1977) used their understanding of series to produce major breakthroughs in number theory and extended the usefulness of series to many areas of mathematics.
History of Vectors

The parallelogram law for the addition of vectors is so intuitive that its origin is unknown. It may have appeared in a now lost work of Aristotle (384--322 B.C.), and it is in the Mechanics of Heron (first century A.D.) of Alexandria. It was also the first corollary in Isaac Newton's (1642--1727) *Principia Mathematica* (1687). In the *Principia*, Newton dealt extensively with what are now considered vectorial entities (e.g., velocity, force), but never the concept of a vector. The systematic study and use of vectors were a 19th and early 20th century phenomenon.

Vectors were born in the first two decades of the 19th century with the geometric representations of complex numbers. Caspar Wessel (1745--1818), Jean Robert Argand (1768--1822), Carl Friedrich Gauss (1777--1855), and at least one or two others conceived of complex numbers as points in the two-dimensional plane, i.e., as two-dimensional vectors. Mathematicians and scientists worked with and applied these new numbers in various ways; for example, Gauss made crucial use of complex numbers to prove the Fundamental Theorem of Algebra (1799). In 1837, William Rowan Hamilton (1805-1865) showed that the complex numbers could be considered abstractly as ordered pairs \((a, b)\) of real numbers. This idea was a part of the campaign of many mathematicians, including Hamilton himself, to search for a way to extend the two-dimensional "numbers" to three dimensions; but no one was able to accomplish this, while preserving the basic algebraic properties of real and complex numbers.

In 1827, August Ferdinand Möbius published a short book, *The Barycentric Calculus*, in which he introduced directed line segments that he denoted by letters of the alphabet, vectors in all but the name. In his study of centers of gravity and projective geometry, Möbius developed an arithmetic of these directed line segments; he added them and he showed how to multiply them by a real number. His interests were elsewhere, however, and no one else bothered to notice the importance of these computations.

After a good deal of frustration, Hamilton was finally inspired to give up the search for such a three-dimensional "number" system and instead he invented a four-dimensional system that he called quaternions. In his own words: October 16, 1843,

> which happened to be a Monday, and a Council day of the Royal Irish Academy – I was walking in to attend and preside, ..., along the Royal Canal, ... an under-current of thought was going on in my mind, which at last gave a result, whereof it is not too much to say that I felt at once the importance. An electric circuit seemed to close; and a spark flashed forth, ... I could not resist the impulse ... to cut with a knife on a stone of Brougham Bridge, as we passed it, the fundamental formulae....

Hamilton's quaternions were written, \(q = w + ix + jy + kz\), where \(w, x, y,\) and \(z\) were real numbers. Hamilton quickly realized that his quaternions consisted of two distinct parts. The first term, which he called the *scalar* and \(x, y, z\) for its three rectangular components, or projections on three rectangular axes, he [referring to himself] has been induced to call the trinomial expression itself, as well as the line which it represents, a VECTOR." Hamilton used his "fundamental formulas," \(i^2 = j^2 = k^2 = -1\), to multiply quaternions, and he immediately discovered that the product, \(q_1q_2 = -q_2q_1\), was not commutative.

Hamilton had been knighted in 1835, and he was a well-known scientist who had done fundamental work in optics and theoretical physics by the time he invented quaternions, so they were given immediate recognition. In turn, he devoted the remaining 22 years of his life to their development and promotion. He wrote two exhaustive books, *Lectures on Quaternions* (1853) and *Elements of Quaternions* (1866), detailing not just the algebra of quaternions but also how they could be used in geometry. At one point, Hamilton wrote, "I still must assert that this discovery appears to me to be as important for the middle of the nineteenth century as the discovery of fluxions was for the close of the seventeenth." He acquired a disciple, Peter Guthrie Tait (1831--1901), who in the 1850s began applying quaternions to problems in electricity and magnetism and to other problems in physics. In the second half of the 19th century, Tait's advocacy of quaternions produced strong reactions, both positive and negative, in the scientific community.

At about the same time that Hamilton discovered quaternions, Hermann Grassmann (1809--1877) was composing *The Calculus of Extension* (1844), now well known by its German title, *Ausdehnungslehre*. In 1832, Grassmann began development of "a new geometric calculus" as part of his study of the theory of tides, and he subsequently used these tools to simplify portions of two classical works, the *Analytical Mechanics* of Joseph Louis Lagrange (1736-1813) and the *Celestial Mechanics* of Pierre Simon Laplace.
In his *Ausdehnungslehre*, first, Grassmann expanded the conception of vectors from the familiar two or three dimensions to an arbitrary number, \( n \), of dimensions; this greatly extended the ideas of space. Second, and even more generally, Grassmann anticipated a good deal of modern matrix and linear algebra and vector and tensor analysis.

Unfortunately, the *Ausdehnungslehre* had two strikes against it. First, it was highly abstract, lacking in explanatory examples and written in an obscure style with an overly complicated notation. Even after he had given it serious study, Möbius was not able to understand it fully. Second, Grassmann was a secondary school teacher without a major scientific reputation (compared to Hamilton). Even though his work was largely ignored, Grassmann promoted its message in the 1840s and 1850s with applications to electrodynamics and to the geometry of curves and surfaces, but without much general success. In 1862, Grassmann published a second and much revised edition of his *Ausdehnungslehre*, but it too was obscurely written and too abstract for the mathematicians of the time, and it met essentially the same fate as his first edition. In the later years of his life, Grassmann turned away from mathematics and launched a second and very successful research career in phonetics and comparative linguistics. Finally, in the late 1860s and 1870s, the *Ausdehnungslehre* slowly began to be understood and appreciated, and Grassmann began receiving some favorable recognition for his visionary mathematics. A third edition of the *Ausdehnungslehre* was published in 1878, the year after Grassmann died.

During the middle of the nineteenth century, Benjamin Peirce (1809--1880) was far and away the most prominent mathematician in the United States, and he referred to Hamilton as, "the monumental author of quaternions." Peirce was a professor of mathematics and astronomy at Harvard from 1833 to 1880, and he wrote a massive *System of Analytical Mechanics* (1855; second edition 1872), in which, surprisingly, he did not include quaternions. Rather, Peirce expanded on what he called "this wonderful algebra of space" in composing his *Linear Associative Algebra* (1870), a work of totally abstract algebra. Reportedly, quaternions had been Peirce’s favorite subject, and he had several students who went on to become mathematicians and who wrote a good number of books and papers on the subject.

James Clerk Maxwell (1831--1879) was a discerning and critical proponent of quaternions. Maxwell and Tait were Scottish and had studied together in Edinburgh and at Cambridge University, and they shared interests in mathematical physics. In what he called "the mathematical classification of physical quantities," Maxwell divided the variables of physics into two categories, scalars and vectors. Then, in terms of this stratification, he pointed out that using quaternions made transparent the mathematical analogies in physics that had been discovered by Lord Kelvin (Sir William Thomson, 1824--1907) between the flow of heat and the distribution of electrostatic forces. However, in his papers, and especially in his very influential *Treatise on Electricity and Magnetism* (1873), Maxwell emphasized the importance of what he described as "quaternion ideas ... or the doctrine of Vectors" as a "mathematical method ... a method of thinking." At the same time, he pointed out the inhomogeneous nature of the product of quaternions, and he warned scientists away from using "quaternion methods" with its details involving the three vector components. Essentially, Maxwell was suggesting a purely vectorial analysis.

William Kingdon Clifford (1845--1879) expressed "profound admiration" for Grassmann's *Ausdehnungslehre* and clearly favored vectors, which he often called steps, over quaternions. In his *Elements of Dynamic* (1878), Clifford broke down the product of two quaternions into two very different vector products, which he called the scalar product (now known as the dot product) and the vector product (today we call it the cross product). For vector analysis, he asserted "[M]y conviction [is] that its principles will exert a vast influence upon the future of mathematical science." Though the *Elements of Dynamic* was supposed to have been the first of a sequence of textbooks, Clifford never had the opportunity to pursue these ideas because he died quite young.

The development of the algebra of vectors and of vector analysis as we know it today was first revealed in sets of remarkable notes made by J. Willard Gibbs (1839--1903) for his students at Yale University. Gibbs was a native of New Haven, Connecticut (his father had also been a professor at Yale), and his main scientific accomplishments were in physics, namely thermodynamics. Maxwell strongly supported Gibbs’s work in thermodynamics, especially the geometric presentations of Gibbs's results. Gibbs was introduced to quaternions when he read Maxwell's *Treatise on Electricity and Magnetism*, and Gibbs also studied Grassmann's *Ausdehnungslehre*. He concluded that vectors would provide a more efficient tool for his work in physics. So, beginning in 1881, Gibbs privately printed notes on vector analysis for his students, which were widely distributed to scholars in the United States, Britain, and Europe. The first book on modern vector analysis in English was *Vector Analysis* (1901), Gibbs's notes as assembled by one of his last graduate students, Edwin B. Wilson (1879--1964). Ironically, Wilson received his undergraduate education at Harvard (B.A. 1899) where he had learned about quaternions from his professor, James Mills Peirce (1834--1906), one of Benjamin Peirce's sons. The Gibbs/Wilson book was reprinted in a paperback edition in 1960. Another contribution to the modern understanding and use of vectors was made by Jean Frenet (1816--1900). Frenet entered École normale supérieure in 1840, then studied at Toulouse where he
wrote his doctoral thesis in 1847. Frenet's thesis contains the theory of space curves and contains the formulas known as the Frenet-Serret formulas (the TNB frame). Frenet gave only six formulas while Serret gave nine. Frenet published this information in the *Journal de mathematique pures et appliques* in 1852.

In the 1890s and the first decade of the twentieth century, Tait and a few others derided vectors and defended quaternions while numerous other scientists and mathematicians designed their own vector methods. Oliver Heaviside (1850--1925), a self-educated physicist who was greatly influenced by Maxwell, published papers and his *Electromagnetic Theory* (three volumes, 1893, 1899, 1912) in which he attacked quaternions and developed his own vector analysis. Heaviside had received copies of Gibbs's notes and he spoke very highly of them. In introducing Maxwell's theories of electricity and magnetism into Germany (1894), vector methods were advocated and several books on vector analysis in German followed. Vector methods were introduced into Italy (1887, 1888, 1897), Russia (1907), and the Netherlands (1903). Vectors are now the modern language of a great deal of physics and applied mathematics and they continue to hold their own intrinsic mathematical interest.
History of Multivariable Calculus

During the 16th century, mathematicians were developing new mathematics to solve problems in physical science. Because the physical world is multidimensional (i.e., three space dimensions and time), many of the quantities used in these applied models were multivariable. Astronomy was one area of science that was rich in this kind of multivariable mathematics. Therefore, the stage was being set by astronomers and mathematicians for the development of multivariable functions and eventually multivariable calculus. Galileo (1564--1642) attempted to apply mathematics to his work in astronomy and to the physics of kinematics and strength of materials. For his fundamental work in these areas, he is often called the founder of modern mechanics and physics. The German astronomer, mathematician, and physicist Johannes Kepler (1571--1630) contributed greatly through the development of his three laws of planetary motion. These results changed astronomy and played a crucial role in the development of Newtonian physics and calculus. His work helped discredit Ptolemy’s geocentric model and helped establish Copernicus’s heliocentric theory. It also set the stage for the rise of multivariable applied mathematics.

After the development of single-variable calculus in the 17th century, its application to problem solving in a multidimensional world resulted in the need for generalization to include functions of more than one variable and multivariable calculus. What were the analogs of the derivative and the integral for functions of more than one variable? Jean d'Alembert (1717--1783) developed and used multivariable calculus in dealing with methods of solving differential equations and with the motion of bodies in a resisting medium. In several ways, he used the works of Newton, L'Hospital, and the Bernoullis to extend the concepts of calculus to several variables. D'Alembert performed research in this area and published many works on mathematics and mathematical physics. His major work was Traité de dynamique (1743), which helped make partial differentiation part of calculus.

Next in line in refining and using multivariable calculus was Joseph Louis Lagrange (1736--1813). He applied his knowledge of calculus to mechanics. He was very productive in this applied area of mathematics. His major works were in the equations of motion and the understanding of potential energy. Lagrange was also the first to develop today's methods of finding maxima and minima using calculus. His work in multivariable optimization resulted in the technique we now call Lagrange multipliers. He was only nineteen when he devised these methods, and even much later in his life he still regarded them as his best work in mathematics. He published Mécanique analytique (1787), which applied multivariable calculus to the motion and properties of objects in space. Lagrange's colleague, astronomer and mathematician Pierre-Simon Laplace (1749--1827), made an impact at a young age by solving a problem of mutual gravitation that had eluded both Euler and Lagrange. His work contributed to the analysis of the stability of the solar system. Laplace generalized the laws of mechanics for their application to the motion and properties of the heavenly bodies, therefore needing and developing results in multivariable calculus. His famous treatise on this subject was entitled Mécanique celeste. In 1782, Adrien Legendre (1752--1833) won a research prize from the Berlin Academy for his work on exterior ballistics. He analyzed the curve described by the flight of cannonballs, taking into consideration the resistance of the air and developing relations for ranges given initial velocities. Legendre was able to develop these equations from his advanced work in differential equations and multivariable calculus.

Sylvestre François Lacroix (1765--1843) wrote an important treatise on calculus in 1797. In this book, he united and generalized many methods to include multivariable results. While Lacroix followed many of the foundations established by Euler, he also incorporated results made later in the 18th century in his text. His treatise expanded the role of multivariable calculus in the sciences. Fellow French mathematician Joseph Fourier (1768--1830) also applied calculus to solving practical problems in science. For his expertise, he was selected by Napoleon to go on the Egyptian expedition as a technical advisor for engineering and technical research. Later, Fourier continued his

Foremost among the contributors to results in multivariable calculus was Carl Friedrich Gauss (1777-1855). Gauss’s accomplishments in science and mathematics were astonishing. His development of a theory of planetary orbits was published in 1809. Gauss developed and proved the divergence theorem while working on the theory of gravitation, but his notebooks were not published until many years later; therefore, others were given credit for the development and proof of this important multivariable result. The theorem is sometimes called Gauss’s Theorem. Gauss developed results that established potential theory as a coherent branch of mathematics. Mikhail Ostrogradsky (1801-1862) was the first to publish a proof for the divergence theorem. Ostrogradsky left Russia for Paris in 1822 where he met Laplace, Legendre, Fourier, Poisson, and Cauchy. While working on the theory of heat in the mid-1820s, he formulated the divergence theorem as a tool for turning volume integrals into surface integrals.

French mathematician Siméon Poisson (1781-1840) studied with Lagrange and Laplace, doing his early work in mechanics. He applied mathematics to the applications of elasticity and vibrations. Famous mathematician Augustin Cauchy (1789-1857) built off the multivariable concepts in Laplace’s *Mécanique céleste* and Lagrange’s *Traité des fonctions analytiques*. In 1816 he solved a hydrodynamics problem on the propagation of waves on the surface of a liquid. He used his understanding of partial differentiation and line integrals to analyze solutions and properties of partial differential equations. Applied mathematician Carl Jacobi (1804-1851) developed the theory of determinants and transformations into a powerful tool for evaluating multiple integrals. He also applied transformation methods to study integrals like the ones that arise in the calculation of arc length.

The work of George Green (1793-1841) on the mathematical foundations of gravitation, electricity, and magnetism was published in 1828 in a short book entitled *An Essay on the Application of Mathematical Analysis to Electricity and Magnetism*. Green’s mathematics provided the foundation on which Thomson, Stokes, Rayleigh, Maxwell, and others built the present-day theory of electromagnetism. Central to this development was the multivariable calculus result we now call Green’s Theorem. George Stokes (1819-1903) applied multivariable calculus to the study of hydrodynamics, elasticity, light, gravity, sound, heat, meteorology, and solar physics. He was not the first to develop the integral theorem that we now call Stokes’ Theorem; he learned of it from Thomson in 1850 and a few years later included it among questions on an examination. It has been known as Stokes’ Theorem ever since.

Bernhard Riemann (1826-1866) worked with physicist Wilhelm Weber (1804-1891), introducing the foundational ideas of differential geometry. He went on to study and contribute to multivariable calculus by applying results in dynamics and computational physics. Similarly, Josiah Willard Gibbs (1839-1903), who was born in Connecticut and attended Yale, worked on applied science problems in mathematical physics. His contributions were in thermodynamics, electromagnetics, and mechanics. Sonya Kovalevsky (1850-1891), the most widely known Russian mathematician of the late 19th century, learned mathematics by reading the wallpaper on a bedroom wall that consisted of pages of a mathematics text of Ostrogradsky’s differential and integral calculus. She primarily worked on the theory of partial differential equations.

Further work by scientists and applied mathematicians in the late 19th and 20th centuries refined earlier multivariable results and utilized these techniques in various areas of science and engineering. Physicist James Maxwell (1831-1879) used multivariable tools like the divergence, curl, flux, and potential to advance understanding of optics, light, electricity, and magnetism. Ernst Mach (1838-1916) used many of these same tools to produce new results in mechanics, thermodynamics, and physics. His work influenced Albert Einstein’s later work in relativity and physics. Italian
mathematician Guido Fubini (1879–1943) advanced both the applied and theoretical aspects of multivariable calculus. His work applied mathematics to science to engineering. He proved the method of evaluating iterated integrals that is named for him and utilized the results in mechanics and physics.
Abel, Niels Henrik (1802 - 1829)

Abel was the most famous Norwegian mathematician. He read works of Newton, d'Alembert, Lagrange, Laplace, and Euler and became motivated to study mathematics. He attended the University of Christiania and graduated in 1822. During his schooling, he began working on finding a general solution procedure for the roots of cubic equations. He published papers on solving algebraic and integral equations. In 1824, he proved the impossibility of solving general cubic equations.

Abel traveled to Berlin and worked with Crelle, who published some of Abel's works and supported his research. Abel also traveled to Paris and met with Cauchy. Cauchy was not very receptive of Abel's work, so Abel returned to Norway. There he suffered from illness, poverty, and considerable personal debt. He died young and never realized the potential that he showed with his brilliant work on functions and solving equations.
Albert of Saxony (ca. 1316 - 1390)

Albert's family, called Ricmestorp, were wealthy land owners. Albert attended the University of Paris and achieved renown as a teacher on the faculty of arts at the University. Albert’s writing, which was composed during the years when he was teaching at Paris, consisted mostly of books of issues and questions on Aristotle’s treatises and on some of his own thinking on logic and other mathematical subjects. He wrote on squaring the circle and other geometric problems. He also published books on physics and mechanics, Tractatus proportionum being the most popular and famous. At Paris, he met and worked with fellow mathematician Oresme. Albert worked on church-related affairs for Pope Urban V, eventually being appointed a bishop, which ended his career as a mathematician.

Major publication: Tractatus proportionum
Archimedes (287 - 212 B.C.)

Archimedes was the greatest mathematician of classical times in the west, and some would say the greatest mathematician in history. He was a native of Syracuse on the island of Sicily, and at some time in his early life he visited the Greek center of learning, Alexandria. Here, he made life-long friends with successors of Euclid at the Academy. After Archimedes returned to Syracuse, he established scientific correspondence with these colleagues wherein he shared his scientific achievements. In a famous incident during the siege of Syracuse, a Roman soldier killed Archimedes as the famous scientist was attempting to finish a mathematical problem.

Archimedes' work can be broken down into three greatly overlapping categories: geometry; physics and mechanics; and engineering devices. Archimedes' greatest legacy was in geometry, wherein he stated and rigorously proved theorems that determined the areas of certain plane regions bounded by curves and areas of certain three-dimensional areas; these are called quadature problems. Similarly, Archimedes established the volumes of certain three-dimensional solids bounded by curved surfaces, and these are called cubatures. His most famous quadrature result is that the area under an arbitrary parabolic sector is equal to four-thirds of the area of the largest inscribed triangle. As a bonus in this work, Archimedes established and made definitive the sum of a geometric series. Archimedes derived a very accurate approximation to the area of a circle, and this was equivalent to a very good approximation of \( \pi \). His work combined great imagination and creativity with tremendous precision. He considered his greatest scientific achievement to be the proof that the volume of a sphere is two-thirds of the volume of the circumscribed cylinder. With each of these area and volume results, Archimedes anticipated the integral calculus. However, the development of the integral calculus had to wait until the seventeenth century of our modern era, and especially the work of Newton and Leibniz. In physics and mechanics, Archimedes formulated the law of the lever, showed the importance of the concept of the center of gravity and how to determine it for many objects, and founded the subject of hydrostatics. While Archimedes' place in history rests on his contributions to mathematics and physics, during his lifetime his reputation was based on the utility and value of the mechanical devices he invented. These included the compound pulley, the water screw to pump water for irrigation, optical devices and mirrors, and various war machines, such as fortifications, catapults, and burning mirrors. At his request, Archimedes' tomb was engraved with the figures of a sphere and the cylinder that circumscribes that sphere.

Major theorems: Area of a circle; sum of a geometric series

Quotations:

"There are things which seem incredible to most men who have not studied mathematics."

"Give me a place to stand and I will move the earth."

"Mathematics reveals its secrets only to those who approach it with pure love, for its own beauty."
Aristotle (384 - 322 B.C.)

Aristotle was born in Stagira, a Greek colony. He went to Athens and entered Plato's Academy at age 17. When Plato died in 347, Aristotle left Athens for 12 years. He returned in 335 B.C. when Athens came under Macedonian rule, and had 12 more years of teaching and research there. He set up a school, much like our graduate schools, in Athens called the Lyceum.

The starting point for his scientific contributions was the years he spent in the Academy. The Academy that Aristotle joined in 367 B.C. was distinguished from others in Athens by its interests in mathematics. Aristotle believed that mathematics was an axiomatic science where theorems are derived from basic principles. As such, its hypotheses and definitions are general in nature and have application in more than one problem or system. He adapted and enlarged his model for mathematics to include the physical sciences as well. To Aristotle, mathematics was a science concerning itself with the physical world. With an emphasis on logic, Aristotle made contributions in many areas, including astronomy, biology, physics, politics, and ethics.

Major theorem: Irrationality of the square root of 2

Quotations:

"Education is the best provision for old age."

"The chief forms of beauty are order and symmetry and precision which the mathematical sciences demonstrate in a special degree."
Bernoulli, Daniel (1700 - 1789)

Daniel Bernoulli was the second son of mathematician Johann Bernoulli. In 1713 Daniel began to study logic. During his youth, he was taught mathematics by his father. In 1724 Bernoulli published his *Exercitationes mathematicae* which attracted considerable attention. This publication led to a position at the St. Petersburg Academy. At St. Petersburg, Bernoulli was creative and productive as a scientist. He published several books and articles on mathematics and mechanics. His principal work, *Hydrodynamica* was completed in 1734 but was not published until 1738.

Major publications: *Exercitationes mathematicae, Hydrodynamica*
Bernoulli, Jakob (1654 - 1705)

Jakob Bernoulli was born in Switzerland and received his degree in 1671 after studying philosophy and theology at the request of his father and mathematics and astronomy against the will of his father. His motto became *Invito patre sidera verso* ("Against my father's will, I study the stars") as he began to investigate mathematics and astronomy on his own. His educational pursuits took him to the Netherlands, where he met mathematician Jan Hudde, and to England, where he met Robert Boyle and Robert Hooke. The result of these journeys was his theory of the movement of comets and a theory of gravity. As a result of this work, Bernoulli contributed articles on algebraic subjects to the *Acta eruditorum*.

By working on problems in optics and mechanics, Bernoulli contributed to important developments in infinitesimal geometry and calculus. Bernoulli showed his mastery of the calculus with his analysis of the solutions given by Huygens in 1687 and by Leibniz in 1689 to the problem of the curve of constant descent in a gravitational field. It was in that analysis that he used the term integral. He also studied the catenary, the function that determines the shape of a suspended string or chain. He made use of polar coordinates in several applied problems that he solved. Unfortunately, Jakob had a strained relationship with his younger brother, mathematician Johann Bernoulli. Jakob Bernoulli taught at Basel from 1683 until his death. He was the first mathematician of the Bernoulli family, which became the most famous family in the history of mathematics.

Quotation:

> Even as the finite encloses an infinite series,  
> And the unlimited limits appear,  
> So the soul of immensity dwells in minuta  
> And the narrowest limits, no limit inhere.  
> What joy to discern the minute infinity!  
> The vast to perceive in the small, what Divinity!
Bernoulli, Johann (1667 - 1748)

Johann Bernoulli was born in Switzerland and attended the University of Basel. His doctoral dissertation was in mathematics despite its medical title, which was used to hide his mathematical work from his father who wanted Johann to become a doctor. Bernoulli privately studied mathematics with his gifted brother Jakob, who served in the mathematics chair at the University of Basel. From that time on, both brothers were engrossed in infinitesimal mathematics and were the first to achieve a full understanding of Leibniz's presentation of differential calculus.

The Bernoulli brothers sometimes worked on the same problems, which was unfortunate in view of their jealous and touchy dispositions. In 1691 Bernoulli was in Paris where he presented and defended the new Leibniz calculus. During this period he also met L'Hospital, then France's most famous mathematician. L'Hospital engaged Bernoulli to instruct him in the new calculus. Johann was appointed professor of mathematics at the University of Groningen in Holland in 1695. L'Hospital had Bernoulli continue this instruction by correspondence after Bernoulli left for Holland and later moved back to Basel. Immediately after Jakob Bernoulli’s death, Johann succeeded his brother in the chair at Basel. Bernoulli's criticism of Taylor’s methodus uncrementorum was an attack upon the method of fluxions, as Bernoulli became involved in the priority dispute between Leibniz and Newton. After Newton’s death in 1727, Bernoulli was unchallenged as the leading mathematician in Europe. Bernoulli taught his successor to that title as well, when he instructed Leonhard Euler at the University of Basel. Johann’s son was mathematician Daniel Bernoulli, who also quarreled with Johann over mathematical issues.

Major theorems: L'Hospital's rule, Taylor series

Quotation:

"A quantity which is increased or decreased by an infinitely small quantity is neither increased or decreased."
Birkhoff, George David (1884 - 1944)

Birkhoff attended Harvard and the University of Chicago. He received his PhD from Chicago in 1907 for his dissertation on differential equations. Most of his work was done at Harvard, where he became a full professor in 1919. His major contributions were in dynamical systems (difference equations) and differential equations. He extended the work of Poincaré. He also worked on the four-color problem (colors required to produce a map) and applying mathematics to aesthetics in art, poetry, and music.
Bolzano, Bernhard (1781 - 1848)

Bolzano was born in Prague, Czechoslovakia. From 1791 to 1796 Bolzano was a pupil in the Piarist Gymnasium. Later, he studied at the University of Prague, where he took courses in philosophy, physics, and mathematics. His interest in mathematics was stimulated by reading Kastner's *Anfangsgründe der Mathematik*. Later he returned to Prague where he continued his mathematical studies until his death.

He was attracted to the methodology of science and mathematics, especially calculus and logic. In 1804 he published his geometrical work, some early non-Euclidean ideas, in *Betrachtungen über einige Gegenstände der Elementargeometrie*. Bolzano helped develop results in the calculus and was a believer in Lagrange's method of using *Taylor series* as the basis for calculus. His work in calculus and analysis was published in *Der binomische Lehrsatz* in 1816 and *Rein analytischer Beweis* in 1817.

Major publications: *Betrachtungen über einige Gegenstände der Elementargeometrie*; *Der binomische Lehrsatz*; and *Rein analytischer Beweis*
Cauchy, Augustin-Louis (1789 - 1857)

Cauchy was born in Paris the year the French revolution began. He enjoyed the benefits of an excellent education. As a young boy, he became acquainted with several famous scientists. Laplace was his neighbor, and Lagrange was an admirer and supporter. After completing his elementary education at home, Cauchy attended the École centrale. After months of preparation, he was admitted to the École polytechnique in 1805 to study engineering. Cauchy had already read Laplace’s *Mécanique céleste* and Lagrange’s *Traité des fonctions analytiques*.

In 1811 Cauchy solved a challenging problem set given to him by Lagrange. In 1816 he won a contest of the French Academy on the propagation of waves on the surface of a liquid; the results are now classics in hydrodynamics. He invented the method of characteristics, which is important in the analysis of partial differential equations. In 1816, when Monge and Carnot were expelled from the Académie des sciences, Cauchy was appointed as a replacement member. Over his career, Cauchy was appointed répétiteur, adjunct professor, and finally a full professor at the École polytechnique. His classic works *Cours d’analyse* (*Course on Analysis*, 1821) and *Résumé des leçons ... sur le calcul infinitésimal* (1823) were Cauchy’s greatest contributions to calculus. Cauchy was the first to define fully the ideas of convergence and absolute convergence of *infinite series*. He initiated the rigorous analysis of calculus. He was also the first to develop a systemic theory for complex numbers and to develop the Fourier transform for *differential equations*. During the turbulent political times in France, Cauchy was periodically in exile. He taught at the University of Turin in Switzerland in 1831 to 1833 while in exile from France. He finally settled in the professorship of celestial mechanics at Sorbonne. Cauchy was highly prolific in his publications, writing many articles and books.

Major theorems: Mean value theorem, intermediate value theorem

Major works: *Cours d’analyse*; *Résumé des leçons ... sur le calcul infinitésimal*
Cavalieri, Bonaventura (1598 - 1647)

Cavalieri was born in Milan, Italy. At a young age, Cavalieri began studying geometry. He quickly absorbed the works of Euclid, Archimedes, Apollonius, and Pappus. Later he studied from and worked with Galileo. He was told by Galileo to study calculus. The two mathematicians often corresponded through letters. Cavalieri learned the foundations of calculus and developed his ideas on the method of indivisibles, which was his major contribution to mathematics.

Cavalieri discovered that if two plane regions can be arranged to lie over an interval of the x-axis in such a way that they have identical vertical cross sections at every point, then the regions have the same area. This theorem earned Cavalieri a professorship at the University of Bologna in 1629. Cavalieri was largely responsible for introducing logarithms as a computational tool to the schools of Italy. His other areas of interest included conic sections, trigonometry, astronomy, and optics.
Alexis Clairaut was born in 1713 and died in 1765. He worked as a mathematician during the expedition. He was a mathematical genius who already at the age of twelve had been called to visit the Academy of Sciences in Paris. Like the rest of the expedition members Clairaut also undertook practical chores though in the first place he was there to calculate the measurement results. In a study published in 1743, the Clairaut proposition postulates in a simple way the dependency of the geometrical flattening ratio on the relationship between the gravity and the centrifugal force. Later Clairaut made a mathematical prediction of the appearance of Halley's comet in 1759.
Courant, Richard (1888 - 1972)

Courant obtained a PhD from Göttingen in 1910. He studied under Hilbert and eventually succeeded Klein on the faculty of Göttingen. He founded Göttingen's Mathematics Institute and was its director from 1920 until 1933. His research work was in mathematical physics. Courant left Germany in 1933 and eventually came to New York University. He established an applied mathematics research center at New York University, which now is named the Courant Institute.
Dedekind, Richard (1831 - 1916)

Dedekind grew up in Germany and in 1850 entered the University of Göttingen. There he studied with Bernhard Riemann and Carl Gauss. He began lecturing in probability, and then later began his serious study of number theory. Dedekind became a professor at Brunswick Polytechnic and stayed there for many years. Like Gauss, Dedekind preferred to study the theoretical aspects of number theory. His work on irrational numbers gave the subject a logical foundation. In his later years, Dedekind met and worked with Cantor on the concept of the infinite.
Descartes, René (1596 - 1650)

Frenchman Descartes received an excellent education in law, mathematics, physics, mechanics, and acoustics, first at a Jesuit school and then at the University of Poitiers. He met and developed a lasting relation with the popular mathematician and communicator Father Mersenne. In 1619, Decartes told his colleague Beeckman about a new science that he had developed, which was called analytic geometry. As a young adult, he served in the army of several nations. Later, he lived in Holland where he did most of his mathematical work.

Descartes' *Discours de la méthode* (*Discourse on Method*, 1637) presented several fundamental theories. Its historical foundations rested in the classical texts of Pappus and Diophantus. Descartes sought a symbolic algebra where problems of any sort could be analyzed and classified in terms of the techniques required for their solution. The appendix of this book was entitled *La géométrie*. As a philosopher, Descartes tried to purify algebra by separating the theory from the techniques and applications. Descartes' work was an important step in the development of calculus. The Cartesian coordinate system is named in honor of Descartes. Legend tells us that he thought of this coordinate system while watching a fly crawl around on the ceiling. He noticed that the path of the fly could be described by the fly’s distances from each of the walls. His discovery of analytic geometry was hailed as one of the most remarkable feats in mathematical history.

Major theorem: Rule of signs

Major publications: *Discours de la méthode* (1637); *La géométrie*

Quotations:

"When it is not in our power to follow what is true, we ought to follow what is most probable."

"Very subtle inventions are to be found in mathematics, which are able to delight inquiring minds, to further all the arts, and to reduce the labor of man."

"Hence we must believe that all the sciences are so interconnected, that it is much easier to study them all together than to isolate one from all the others. If, therefore, anyone wishes to search out the truth of things in serious earnest, he ought not select one special science, for all the sciences are cojoined with each other and interdependent."

"I think, therefore I am."
Dirichlet, Johan Peter Gastav Lejeune (1805 - 1859)

Dirichlet was a German mathematician who studied at a Jesuit college and then at the Collège de France. He became a tutor for the royalty of France and met many famous mathematicians. Although Dirichlet's main interest was in number theory, he also contributed in calculus and physics. Dirichlet returned to Germany in 1826 and taught at the military academy and the University of Berlin. He was good friends with colleague Carl Jacobi. Dirichlet investigated the solution and equilibrium of systems of differential equations and discovered many results on the convergence of series. In 1855, Dirichlet succeeded Gauss as the professor of mathematics at Göttingen.
Euclid (ca. 365 - ca. 300 B.C.)

Euclid lived in Alexandria, Egypt and was the most talented and influential mathematician of his time. He was younger than Plato and Aristotile, but older than Archimedes. While he was probably educated in Athens, he taught at the Museum in Alexandria, a research institute stressing science and literature. Euclid recorded, collated, and extended the mathematics of the ancient world. He was one of the most influential mathematicians of all time and a prolific author.

Euclid is best known for his foundational work in geometry, which was presented in the classic work entitled *The Elements*. This work laid the foundation for the subject of geometry and in general for axiomatic mathematics. All of the facts of the subject (e.g., geometry) must be proven deductively as statements of theorems and propositions. The reasoning can depend only on the assumptions made at the start (i.e., the definitions and axioms) and on relevant previously established theorems and propositions. *The Elements* contains 13 books and begins with definitions and axioms, including the famous parallel postulate, which states that one and only one line can be drawn through a given point parallel to a given line. This postulate is a defining component for Euclidean geometry. The books have been translated into many languages and used as a text in mathematics for over two thousand years. In addition to the *Elements*, Euclid wrote other works on geometry, including the theory of conics, and on astronomy, optics, and music; many of these works have been lost. Euclid has given his name to several mathematical concepts, including the Euclidean algorithm. He has been called the Father of Geometry. When asked if there was an easier way to learn geometry than *The Elements*, Euclid replied, "There is no royal road to geometry." *The Elements* was a compilation of the most important theoretical mathematics known at that time and thus probably contained the earlier work and ideas of Pythagoras, Hippocrates, Plato, Aristotile, and Eudoxus; its crowning achievement, however, is the great systematic thinking of Euclid.

Major Theorem: Infinitude of primes

Major work: *The Elements*

Quotations:

"The laws of nature are but the mathematical thoughts of God."

"There is no royal road to geometry."
Euler, Leonhard (1707 - 1783)

Born in Basel, Switzerland, Leonhard Euler was the dominant mathematical figure of his century and the most prolific mathematician who ever lived. He was also an astronomer, physicist, engineer, and chemist. He was the first scientist to give the function concept prominence in his work, thereby setting a strong foundation for the development of calculus and other areas of mathematics. Euler's collected books and papers (over 870 articles and books) fill over 80 volumes. He made tremendous contributions to analytic geometry, trigonometry, calculus, and number theory.

As a young man, Euler showed great promise as a mathematician although his father preferred that he study theology. Fortunately, Johann Bernoulli persuaded Euler's father to allow Euler to concentrate on mathematics. Euler graduated from the University of Basel, where his thesis compared the work of Descartes with that of Newton. Euler took a position in St. Petersburg and for a few years was a medic in the Russian navy. In 1733, he became the professor of mathematics at the St. Petersburg Academy of Sciences. He published the two-volume *Mechanica* in 1736, where he systematically applied calculus to the mechanics of a point mass and incorporated many new differential equations into mechanics. In 1738 he lost sight in his right eye. In 1741, Euler moved to a position as mathematics director of the Berlin Academy of Sciences. His work there included translating and enhancing Robin's *Principles of Gunnery*, publishing *Scientia navalis* in 1749 and *Letters to a German Princess* from 1768 to 1772, and teaching Lagrange by correspondence. In 1766, Euler returned to Russia at the invitation of Catherine the Great. In 1771, he lost sight in his left eye, leaving him totally blind. His work moved to calculus and analysis as he published his trilogy, *Introductio in analysin infinitorum*, *Institutiones calculi differentialis*, and *Institutiones calculi integralis*. These works in a total of six volumes made the function a central part of calculus and built on the subjects of algebra, trigonometry, analytic geometry, and number theory. Through these treatises, Euler had a deep influence on the teaching of mathematics. It has been said that all calculus textbooks since 1748 are essentially copies of Euler or copies of copies of Euler. Some of his contributions to differential equations are: the reduction of order, integrating factor, undetermined coefficients, the theory of second order linear equations, and power series solutions. He also incorporated vector calculus and differential equations into his works.

Euler did for modern analytic geometry and trigonometry what the *Elements* of Euclid had done for geometry, and the resulting tendency to render mathematics and physics in mathematical terms has continued ever since. Euler enriched mathematics with many concepts, techniques, and notations currently in use. Euler brought order to the chaos of mathematical notation. He established most of the notation we use today (sine, cosine, e, pi, i, sigma, f for function). Euler's contributions to number theory and physics were equally impressive. In his work *Theoria motus corporum solidorum seu rigidorum* (*Theory of the Motions of Rigid Bodies*, 1765), he set the foundations of continuum mechanics and lunar theory. His influence in mathematical physics was so pervasive that most of his discoveries are not credited to him. However, we do have Euler equations for rotation of a rigid body, flow of an ideal incompressible fluid, bending of elastic beams, and critical loads for buckling of columns. "He calculated without apparent effort, as men breathe, or as eagles sustain themselves in the wind." Euler was the Shakespeare of mathematics—universal, richly detailed, and inexhaustible.

Major theorems: Summation of series, Königsburg bridge theorem

Major publications: *Introductio in analysin infinitorum*; *Institutiones calculi differentialis*; *Institutiones calculi integralis*; *Theoria motus corporum solidorum seu rigidorum*; *Mechanica*; *Letters to a German Princess*
Fermat, Pierre de (1601 - 1665)

Fermat was born to a prosperous family in France. He studied the classics and mastered Latin, Greek, Italian, and Spanish. One of the seventeenth century’s greatest mathematicians, Fermat hesitated to publish his work and rarely wrote complete descriptions even for his own use. Most of his work was reported in correspondence with fellow mathematicians, Gassendi, Huygens, and Mersenne. Fermat was one of the co-founders, along with Descartes, of analytic geometry. He benefited from reading Viète’s works. Fermat’s book *Ad locos planos et solidos isagoge* (*Introduction to Plane and Solid Loci*) contained a more direct and clearer system than Descartes’ *La géométrie*.

Fermat is probably most famous for his work in number theory. His famous unproved “last theorem” (that $a^n = b^n + c^n$ has no positive integer solutions for $a$, $b$, and $c$ if $n>2$) is known from a note he jotted in the margin of a book. After many talented mathematicians over the period of hundreds of years failed to prove it, this famous theorem was recently proved by Andrew Wiles of Princeton. Fermat’s name slipped into relative obscurity until the late 1800s, and it was from an edition of his works published at the turn of the century that the true importance of his many achievements became clear. Besides his work in physics and number theory, Fermat realized the concept that the area under a curve could be viewed as the limit of sums of rectangle areas (as we do today) and also developed a method for finding the centroids of shapes bounded by curves in the plane. The standard formula for calculating arc length and finding the area of a surface of revolution and the second derivative test for extreme values of functions are also found in his papers. He wrote over 3000 mathematical papers and notes; however, he published only one. He studied maximum and minimum values of functions in advance of differential calculus, and wrote an unpublished account of the conic sections. Fermat made numerous significant contributions to so many branches of mathematics that he has often been called the greatest French mathematician of the seventeenth century. More than 13 years before Newton was born, Fermat had discovered a method for drawing tangents to curves and finding maxima and minima. For all his work in these areas, some mathematicians and historians give Fermat credit for the development of differential calculus. Through correspondence with Pascal, Fermat also helped create the foundation of probability theory.

Major publication: *Ad locos planos et solidos isagoge*

Quotation:

"I have found a very great number of exceedingly beautiful theorems."
Fontenelle, Bernard le Bouyer (1657 - 1757)

Frenchman Fontenelle attended a Jesuit school and after graduation became a lawyer for a short while. He wrote poetry and literature for several years before studying mathematics and performing scientific and mathematical research. He wrote *Entretiens sur la pluralité des mondes* in 1686. This book popularized results in astronomy. He then wrote a number of volumes on the history of science. His work *Élémens de la géométrie de l'infini* in 1727 presented his mathematical theories.

Major publications: *Entretiens sur la pluralité des mondes*; *Élémens de la géométrie de l'infini*
Fourier, Joseph (1768 - 1830)

Fourier was a French mathematician, who as a young boy aspired to become an Army officer. He was denied that opportunity for military service and, therefore, switched his passion to mathematics. After building his reputation as a mathematical scholar, he became involved in the political turmoil of the French revolution. He was imprisoned but then released to attend the École normale and later to teach at the École polytechnique, as a teaching aid to Monge and Lagrange.

Monge chose Fourier to accompany Napoleon's Egyptian expedition as a technical advisor for engineering and technical research. He became a friend of Napoleon. Fourier spent a number of years directing public improvements for the Napoleonic government and publishing the findings of the Egyptian expedition. Somehow, Fourier continued his mathematical research and made contributions in the study and computation of heat diffusion and the solution of differential equations. Much of that work appears in Théorie analytique de la chaleur (The Analytical Theory of Heat, 1822), in which he made extensive use of the series that bear his name. However, he contributed nothing to the mathematical theory of these series, which were well known much earlier to Euler, Daniel Bernoulli, and Lagrange. Fourier received support for his work from Laplace, despite receiving criticism for that work from Poisson.

Major theorem: Fourier series

Major publication: Théorie analytique de la chaleur

Quotation:

"Nature is the most fertile source of mathematical discoveries."
Fubini, Guido (1879 - 1943)

Fubini attended secondary school in Venice, Italy, where he showed that he was brilliant at mathematics. His advanced study was at the Scuola Normale Superiore di Pisa, where his doctoral thesis was in geometry. He then worked on harmonic functions in curved spaces. Fubini taught at the University of Catania in Sicily and later at the University of Genoa and the University of Turin. Fubini’s interests were wide ranging, from differential geometry to analysis and to the applications of differential equations. During World War I Fubini applied his work to benefit the Italian army in the areas of accuracy of artillery, acoustic propagation, and electric circuits. During World War II as Fascism took over Italy, he was forced to retire from his position at Turin and leave Italy. He and his family emigrated to the United States, where he taught for a few years before his death. Fubini’s work in the calculus involved Weierstrass’s integral, surface integrals, and Taylor series. He was said to be one of the most original minds in mathematics during the first half of the 20th century.
Galilei, Galileo (1564 - 1642)

Galileo was an Italian mathematician and astronomer. He attempted to apply mathematics to his work in astronomy, physics of kinematics, and strength of materials. For all his fundamental work in these areas, he is often called the founder of modern mechanics and physics. Without the funds to attend the university at Pisa, Galileo studied mathematics on his own. He read Euclid and Archimedes. Based on his studies of Archimedes, Galileo improved the concepts and results in hydrostatics. Soon he was appointed the professor of mathematics at the University of Pisa and later at the University of Padua. There he developed the results for free-fall motion under gravity and the motion of the planets. His astronomy work was published in his famous book Sidereus nuncius (The Starry Messenger, 1610)

Major publication: Sidereus nuncius

Quotations:

"Where the senses fail us, reason must step in."

"If I were again beginning my studies I would follow the advice of Plato and start with mathematics."

"Infinities and indivisibles transcend our finite understanding, the former on account of their magnitude, the latter because of their smallness; imagine what they are when combined."
Gauss, Carl Friedrich (1777 - 1855)

Gauss was born in Brunswick, Germany. He attended the University of Göttingen. Gauss contributed immensely to both applied and pure mathematics. The list of Gauss's accomplishments in science and mathematics is astonishing, ranging from the invention of the electric telegraph (with Wilhelm Weber in 1833) to the development of a theory of planetary orbits to the development of an accurate theory of non-Euclidean geometry. Gauss was very careful to make all his publications and proofs of theorems perfect. Still he is credited with many results in algebra, number theory, differential equations, and calculus. This major work was entitled *Disquisitiones arithmeticae* (1801). He also published *Theoria motus corporum celestium* (1809).

Gauss served as the professor of mathematics at Göttingen. Gauss's presence automatically made Göttingen the center of the mathematical world. But Gauss was remote and unapproachable—particularly to beginning students. He gave the first satisfactory proof of the Fundamental Theorem of Algebra. Gauss's discoveries were so important and numerous that he is often called the Prince of Mathematics. Gauss proved the divergence theorem while working on the theory of gravitation, but his notebooks were not published until many years later; therefore, others were given credit. The theorem is now sometimes called Gauss's Theorem. Gauss established potential theory as a coherent branch of mathematics. He also recognized that theory of functions of a complex variable was the key to understanding many of the results needed in applied differential equations. Gauss considered mathematics a science and arithmetic its most important component.

Major theorems: Divergence theorem

Major publications: *Disquisitiones arithmeticae*, *Theoria motus corporum celestium*

Quotations

"In mathematics there are no true controversies."

"I have the result, but I do not yet know how to get it."
Gibbs, Josiah Willard (1839 - 1903)

Gibbs was born in Connecticut where his father was a literature professor at Yale. Gibbs attended Yale and was better known as a humanities student during his undergraduate days than a mathematician. However, for his graduate work at Yale, he worked on an applied science problem, and his degree is considered the first engineering PhD and second science PhD awarded in America. Gibbs then visited Europe for three years, mostly studying and working in mathematical physics. He returned to Yale to become a professor of mathematics.

He made contributions to thermodynamics, electromagnetics, and statistical mechanics. For his foundational work, Gibbs is known as the father of vector analysis. Aristotle used vectors to describe the effects of forces, and the idea of resolving vectors into geometric components parallel to the coordinate axes came from Descartes. The algebra of vectors we use today was developed simultaneously and independently in the 1870s by Gibbs and by the English mathematical physicist Oliver Heaviside. The works of Gibbs and Heaviside grew out of more complicated mathematical theories developed some years earlier by the Irish mathematician William Hamilton and the German geometer Hermann Grassmann. Hamilton's quaternions and Grassmann's algebraic forms are still in use today but tend to appear in more theoretical work. Vector analysis is used much more frequently and is important in many ways in calculus and other branches of mathematics. In the late 1890s, Gibbs wrote an article that described the convergence and the Gibbs phenomenon of Fourier series.
Grassmann, Hermann (1809 - 1877)

Grassmann was born in Prussia (modern-day Poland) and attended the University of Berlin. However, his study of mathematics and physics was done on his own. He taught at a secondary school and worked on his research without support. In 1844, he published *Die lineale Ausdehnungslehre*, which contained new concepts in geometric calculus. He introduced the $n$-dimensional vector space and new concepts and structures in linear algebra. Unfortunately, Grassmann's book was difficult to read and was largely ignored by other mathematicians. Disappointed by the rejection of his book, Grassmann moved away from mathematics, periodically returning to attempt to revitalize his work. Unfortunately, it was only after Grassmann died that other mathematicians realized the potential and scope of Grassmann's work and from its foundation built the subject of matrix algebra.

Major publication: *Die lineale Ausdehnungslehre*
Gregory, James (1638 - 1675)

Gregory was born in Scotland and graduated from Marischal College, where he focused on mathematics and astronomy. He went to London to seek more opportunities and published *Optica promota* (1663). Next he went to the University of Padua in Italy to study and work. There he published *Vera circuli et hyperbolae quadratura* (1667) and *Geometriae pars universalis, inserviens quantitatum curvarum transmutationi & measurae* (1668). These two works looked at areas of geometric shapes, properties of series, and the difference between algebraic and transcendental functions. The later book also contained a proof of the fundamental theorem of calculus. Gregory returned to London and then Scotland in 1668 and was appointed the professor of mathematics at St. Andrews College in Scotland. He left there to go to Edinburgh University in 1674.

Major publications: *Vera circuli et hyperbolae quadratura; Geometriae pars universalis, inserviens quantitatum curvarum transmutationi & measurae*
Halley, Edmund (1656 - 1742)

Halley, British biologist, geologist, sea captain, astronomer, and mathematician, encouraged Newton to write the *Principia*. Despite all of Halley's accomplishments, he is known today as the man who calculated the orbit of the comet of 1682: "therefore if according to what we have already said [the comet] should return again about the year 1758, candid posterity will not refuse to acknowledge that this was first discovered by an Englishman." Ever since the comet's return in 1758, it has been known as Halley's comet. Last seen rounding the sun during the spring of 1986, the comet is due to pass by earth again in the year 2062. A recent study indicates that the comet has made about 2000 cycles so far with about the same number to go before it erodes away completely.
Heaviside, Oliver (1850 - 1925)

Heaviside studied electricity and languages on his own. He became a telegrapher and began working on results in electricity. He wrote two papers that caught the interest of Maxwell. Thereafter, Maxwell's work inspired Heaviside. Heaviside was able to simplify Maxwell's 20 equations into the two we now call Maxwell's equations. Heaviside's contributions in mathematics were in the areas of vector algebra and vector calculus. Using his methods he was able to efficiently solve systems of differential equations. He worked with Gibbs (in mathematics) and Thomson (in electricity).
In the late 1600s, John Fernoulle discovered a rule for calculating limits of fractions whose numerators and denominators both approach zero. Today the rule is known as l'Hôpital's rule. L'Hôpital was a French nobleman who wrote the introductory calculus text in which the rule first appeared. L'Hôpital's rule often produces fast and direct results and sometimes works when other methods fail. In 1691, Johann Bernoulli agreed to accept a retainer of 300 pounds per year from his former pupil l'Hôpital to solve problems for him and keep him up to date on calculus. One of the problems was the called the 0/0 problem, which Bernoulli solved. When l'Hôpital published his calculus book in 1696, the 0/0 rule appeared as a theorem. L'Hôpital acknowledged his debt to Bernoulli and, to avoid claiming complete authorship, had the book appear without bearing his own name. Nevertheless, Bernoulli accused l'Hôpital of plagiarism for publishing Bernoulli's results in the book.

Major publication: The first calculus book: Analyse des infiniment petits pour l'intelligence des linies courbes
Huygens, Christiaan (1629 - 1695)

Huygens was born in the Hague, Netherlands. He studied mathematics at the University of Leiden. His family was wealthy enough that Huygens was able to perform his mathematical research without additional support or a salary. He traveled around Europe eventually settling in Paris from 1666 to 1680. Huygens was a follower of Descartes. He published his important geometrical results in *Theoremata de quadratura hyperboles, ellipses et circuli* and *De circuli magnitudine inventa* (1654). Later he considered the subject of probability and published *Tractatus de ratiociniis in aleae ludo* (1657).

He tackled the problem that a pendulum clock whose bob swings in a circular arc has a frequency of swing dependent on the amplitude of the swing. The wider the swing, the longer it takes the bob to return to center. This does not happen if the bob can be made to swing in a cycloid. In 1673, Huygens, driven by a need to make accurate determinations of the longitude at sea, designed a pendulum clock whose bob swung in a cycloid. He hung the bob from a fine wire constrained by guards that caused it to draw up as it swung. More important than his clock invention was Huygens's development of a wave theory for light. Huygens's study of optics helped his astronomical work by providing him an improved telescope. Huygens, using his new, improved telescope, discovered the rings of Saturn, which were not distinguishable using earlier telescopes.
Jacobi, Carl Gustav Jacob (1804 - 1851)

Jacobi, one of nineteenth-century Germany’s most accomplished scientists, developed the theory of determinants and transformations into a powerful tool for evaluating multiple integrals and solving differential equations. He also applied transformation methods to study integrals like the ones that arise in the calculation of arc length. Like Euler, Jacobi was a prolific writer, a remarkably skilled calculator, and an expert in a variety of mathematical and applied fields.

Quotation:

"The real end of science is the honor of the human mind."
Kepler, Johannes (1571 - 1630)

The German astronomer, mathematician, and physicist Johannes Kepler was the first scientist to demand physical explanations of celestial phenomena. His three laws of planetary motion, the results of a lifetime of work, changed astronomy and played a crucial role in the development of Newtonian physics and calculus. His work helped discredit Ptolemy's geocentric model and helped establish Copernicus's heliocentric theory. Kepler studied at the University of Tübingen (1589 - 1594) and was a brilliant student.

After graduating, Kepler taught mathematics at Graz. Later, he moved to Prague to assist astronomer Tycho Brahe. Kepler succeeded Brahe and in doing so gained access to Brahe's considerable astronomical observations. Kepler began to construct the Rudolphine Tables, which provided highly accurate planetary observations. Kepler used the data on Mars's orbit to determine that the orbit was an ellipse with the sun at one focus. Kepler's first and second laws (elliptical orbits, equality of areas swept out by motions of planets) were published in his book Astronomia nova (New Astronomy, 1609). Kepler praised the advances made in Galileo's Starry Messenger published a year later. Kepler's Harmonice mundi (Harmonies of the World, 1618) contained his third law (cube:square ratio). Some of Kepler's summations contained in his books are remarkably similar to results discovered later and used in integral calculus.

Major publications: Astronomia nova, Harmonice mundi

Quotations:

"Happy is the man who devotes himself to the study of the heavens ... their study will furnish him with the pursuit of enjoyments."

"Where there is matter, there is geometry."
Kovalevsky, Sonya (1850 - 1891)

Kovalevsky was the greatest woman mathematician prior to the twentieth century. She was the most widely known Russian mathematician in the late 19th century. When she was 14, Kovalevsky began reading the wallpaper on a bedroom wall that consisted of pages of a mathematics text of Ostrogradsky's differential and integral calculus. Her careful study of that wall-covering provided Kovalevsky with an introduction to calculus. In 1865 she took a more rigorous course under the tutelage of Aleksandr Strannolyubsky, mathematics professor at the naval academy in St. Petersburg, who recognized her potential as a mathematician.

Kovalevsky went to Heidelberg in 1869, where she took mathematics courses with Kirchhoff, Helmholtz, Koenigsberger, and du Bois-Reymond. In 1871 she left for Berlin, where she studied with Weierstrass. As a woman, she could not be admitted to university lectures; consequently, Weierstrass tutored her privately during the next four years. By 1874 she had completed three research papers on partial differential equations. In spite of Kovalevsky's doctorate and strong letters of recommendation from Weierstrass, she was unable to obtain an academic position anywhere in Europe. Finally, through the efforts of the Swedish analyst Mittag-Leffler, Kovalevsky was appointed to a lectureship in mathematics at the University of Stockholm. In her study of the form of Saturn's rings, she built off the work of Laplace, whose work she generalized. She primarily worked on the theory of partial differential equations, and a central result on the existence of solutions still bears her name. She published numerous papers on partial differential equations, eventually gaining recognition as the first woman to be elected a member of the Russian Imperial Academy of Sciences in 1889. An unusual aspect of Kovalevsky's life was that, along with her scientific work, she had a simultaneous career in literature and wrote several successful novels. Besides novels about Russian life, she wrote her *Recollections of Childhood*, which has been translated into English.

Quotation:

"It is impossible to be a mathematician without being a poet in soul."
Lagrange, Joseph Louis (1736 - 1813)

Lagrange was born in Turin, Italy. He enjoyed studying mathematics, despite his father's wish that he study law. Lagrange's mathematical contributions began as early as 1754 with the discovery of the calculus of variations and continued with applications to mechanics in 1756. He was very productive in mechanics as well as in differential and integral calculus. Both Euler and d'Alembert praised his work.

Lagrange's mathematical career can be viewed as a natural extension of the work of his older and greater contemporary, Euler, which in many respects he carried forward and refined.

Lagrange's major works were in the equations of motion and the understanding of potential energy. From 1755 to 1766, he served as a professor at the Royal Artillery School in Turin. He also helped to establish the Turin Academy of Sciences. He succeeded Euler as mathematics director of the Berlin Academy in 1766. There he worked on the 3-body problem (analyzing the mutual attraction of 3 large bodies, like planets), number theory, and calculus. Then in 1787, he moved to Paris. Lagrange's years in Paris were dedicated to the composition of the great treatises summarizing his mathematical concepts. He published Mecanique analytique (1788), which applied calculus to the motion of objects. In 1794, he helped Monge establish the Ecole polytechnique, and a year later he taught at the Ecole normale with Laplace. Lagrange's greatest work was in the theory and application of the calculus. He carried forward Euler's work of putting calculus on firm algebraic ground in his theory of functions. He also made many contributions to algebra and number theory. He worked on weights and measures, and in effect was the father of the metric system. Napoleon thought very highly of Lagrange and used and supported his work to the benefit of France and Napoleon's army. Lagrange was the first to develop today's methods of finding maxima and minima. He was only nineteen when he devised these methods and he regarded them as his best work in mathematics. One of Lagrange's favorite computational and theoretic tools was integration by parts. He felt that Taylor series played the fundamental role in understanding calculus.

Major publication: Mecanique analytique

Quotation:

"As long as algebra and geometry proceeded along separate paths, their advance was slow and their applications limited. But when these sciences joined company, they drew from each other fresh vitality and thenceforth marched on at a rapid pace."
Laplace, Pierre-Simon (1749 - 1827)

Mathematician and astronomer, Laplace was born in Normandy, France. He attended the University of Caen and then at age 19 moved to Paris to teach at the École militaire. From 1795 to 1799, he taught at the École polytechnique. Laplace initially made an impact by solving a complex problem of mutual gravitation that had eluded both Euler and Lagrange. Laplace was among the most influential scientists of his time and was called the Newton of France for his study of and contributions to the understanding of the stability of the solar system. Laplace generalized the laws of mechanics for their application to the motion and properties of the heavenly bodies. He is also famous for his great treatises entitled Mécanique céleste (1799 - 1825) and Théorie analytique des probabilités (1812). They were advanced in large part by the mathematical techniques that Laplace developed early in his life; most notably among those techniques are generating functions, differential operators, and definite integrals.

Major publications: Mécanique céleste, Théorie analytique des probabilités

Quotations:

"What we know is very little; what we do not know is immense."

"All the effects of nature are but the mathematical results of a small number of immutable laws."

"Read Euler, read Euler, he is the master of us all."
Legendre, Adrien-Marie (1752 - 1833)

Legendre received an education in mathematics that was unusually advanced for the Paris schools in the eighteenth century. He was fortunate that his primary mathematics teacher was the Abbé Joseph-François Marie, who was a master teacher. Legendre was a quick student and graduated at a young age. Soon thereafter, he was teaching mathematics at the École polytechnique. In 1782 Legendre won a research prize from the Berlin Academy. The subject of the competition that year concerned exterior ballistics: “Determine the curve described by cannonballs and bombs, taking into consideration the resistance of the air, and give rules for obtaining the ranges corresponding to different initial velocities.” Legendre was able to develop these equations from his advanced work in differential equations.

Legendre encountered and developed polynomials, today named for him, in his research on the gravitational attraction of ellipsoids. He devoted 40 years to this research to elliptic integrals.
Leibniz, Gottfried Wilhelm (1646 - 1716)

Leibniz was born in Leipzig, Germany. At a young age, he was given access to the library that his father had assembled. Leibniz thus became acquainted with a wide range of classical writers and began omnivorous reading that was his habit throughout his life. At the age of fifteen, Leibniz entered the University of Leipzig, where he received most of his formal education. His interest in mathematics was aroused by the numerous remarks on the importance of the subject that he encountered in his reading of philosophical works. Later he attended the University of Altdorf, near Nuremberg, where he received a doctorate in law. He practiced law in Paris to support his study of mathematics.

Leibniz summed sequences of reciprocal polygonal numbers and, following the work of St. Vincent, summed and analyzed geometric sequences. He studied trigonometric functions from the works of Huygens. In 1671 Leibniz developed a machine that could not only add and subtract but also multiply, divide, and extract square roots. The Leibniz calculator was gear-operated, and it provided a carry from one order to the next. His machine was the first general-purpose calculator whose principles are still used in mechanical calculating machines. Late in 1675, Leibniz laid the foundations of both integral and differential calculus. In the calculus, Leibniz developed the present-day notation and many computational methods for the derivative and integral. Although he never thought of the derivative as a limit, he discovered many of the results we now study in calculus. Along with Newton, Leibniz is given credit by most sources as the developer of calculus.

Major theorems: Fundamental theorem of calculus; series for $\pi$

Quotations:

"It is unworthy of excellent men to lose hours like slaves in the labor of calculation which could safely be regulated to anyone else if machines were used."

"Taking mathematics from the beginning of the world to the time of Newton, what he has done is much the better half."
Maclaurin, Colin (1698 - 1746)

In 1709 Maclaurin entered the University of Glasgow where he became acquainted with Robert Simson, professor of mathematics. In 1715 Maclaurin defended the thesis *On the Power of Gravity*, for which he was awarded his degree. It led to his appointment, in 1717, as professor of mathematics at Marischal College in Aberdeen, although he was still only in his teens. This appointment marked the beginning of a brilliant mathematical career. His book *Geometrica organica, sive descriptio linearum curvarum universalis* dealt with the general properties of conics and of higher plane curves. Maclaurin was elected a fellow of the Royal Society of London in 1719.

Maclaurin's *Treatise of Fluxions* (1742) has been described as the earliest logical and systematic publication of Newton's methods. It stood as the model of rigor until the appearance of Cauchy's *Cours d'analyse* in 1821. Maclaurin was a zealous disciple of Newton, and in his work he tried to provide a geometrical framework for the doctrine of fluxions. Maclaurin hoped to refute Newton's critics, the most vociferous of whom was George Berkeley.

Major publications: *Geometrica organica, sive descriptio linearum curvarum universalis; Treatise of Fluxions*
Mersenne, Marin (1588 - 1648)

Mersenne was a French priest who played an important role in mathematics. Mersenne corresponded with numerous mathematicians throughout Europe, passing important information between them. There were no scientific journals during his time, so he played the role of a disseminator of valuable information. He frequently met with Fermat, Roberval, and Pascal. His own work was in generating prime numbers and extending Galileo’s work in acoustics. Mersenne actively defended Galileo and Descartes against attacks from the church.

Major publications: *Traité des mouvements*; *Les Méchanique de Galilée*. 
Napier, John (1550 - 1617)

Scotsman John Napier went to St. Salvator's College in St. Andrews, where he studied from mathematician John Rutherford. In sixteenth-century Scotland, intellectual interests centered on religion, theology, and politics rather than on science and mathematics, and Napier's first work reflected that climate. Napier was a fervent Protestant and a landowner of a large estate and farm. There is evidence that Napier began working on the idea of logarithms about 1590. His important mathematical work culminated in the publication of two Latin treatises. In the Constructio, the phrase artificial numbers is used by Napier instead of logarithms. The word logarithm was adopted later.

Today Napier is best known as the inventor of logarithms, but until recently we knew very little about their invention. Today we realize that in the late 1500s, Napier invented a computational tool called the logarithm that simplified arithmetic by replacing multiplication by addition. The equation that accomplished this was simply \( \ln (ax) = \ln a + \ln x \). To multiply two positive numbers \( a \) and \( x \), you looked up their logarithms in a table, added the logarithms, found the sum in the table, and read the table backward to find the product \( ax \). Having such a table was the key, and Napier spent the last 20 years of his life working on a table that he never finished (while astronomer Tycho Brahe waited in vain for a complete table so could speed up his astronomical calculations). The table was completed after Napier's death (and Brahe's) by Napier's friend Henry Briggs in London. Logarithms became a powerful tool in astronomical and navigational computations. Logarithms subsequently became widely known as Briggs' logarithms and some older books on navigation still refer to them by this name. Napier invented a mechanical device in 1617 that was arranged of bone strips on which numbers were stamped out. When placed into the proper combination, "Napier's bones" could perform multiplication. Napier's bones were used by Oughtred in 1630 in the invention of the slide rule. Napier did other mathematical work including spherical trigonometry and development of decimal notation.

Napier's major works are Mirifici logarithmorum canonis descriptio (The Description of the Wonderful Canon of Logarithms, 1614) and Mirifici logarithmorum canonis constructio (The Construction of the Wonderful Canon of Logarithms, 1619).

Major publications: Mirifici logarithmorum canonis descriptio; Mirifici logarithmorum canonis constructio

Quotation:

"Seeing there is nothing that is so troublesome to mathematical practice, nor that doth more molest and hinder calculators, than the multiplications, divisions, square and cubical extractions of great numbers .... I began therefore to consider in my mind by what certain and ready art I might remove those hindrances."
Newton, Isaac (1642 - 1727)

In his youth in England, Newton was interested in mechanical devices and their underlying theories. He actually constructed lanterns and windmills that he designed. Among the books that Newton studied were Kepler's Optics, Euclid's The Elements, and Descartes' La géométrie. He also studied the works of Glanvill, Boyle, and Gassendi's Copernican astronomy, which was then published along with Galileo's Sidereus nuncius and Kepler's Dioptre. Newton went to Trinity College, Cambridge, and received a degree in 1665.

At the age of 26, Newton was appointed Lucasian professor at Cambridge, which required that Newton give at least one lecture a week in every term. He had already developed much of the calculus and many important results in physics. During the years in the 1680s when Newton was writing the Principia, he seldom left his chambers except at term time. He corresponded, both directly and indirectly, with scientists in England and on the European continent, including Boyle, Collins, Flamsteed, David Gregory, Halley, Hooke, Huygens, Leibniz, and Wallis. Halley played a significant role in getting Newton to write the Principia. Newton was a precise scholar with a great intellect and tremendous skill at problem solving. During the 1670s and 1680s, he built his reputation as a scientific genius. Newton was one of the greatest mathematicians of all time. Newton's influence on mathematics was so great that the field is sometimes divided into pre-Newtonian and post-Newtonian mathematics. Newton's contributions included the theory of universal gravitation, the laws of motion, methods of calculus, and the composition of white light. He is considered by many to be the greatest scientist who ever lived.

Major theorem: Binomial theorem

Major publication: Principia (Full title: Philosophiae naturalis principia mathematica)

Quotations:

"If I have seen farther, it was because I stood on the shoulders of giants."

"It is the glory of geometry that from so few principles, ... it is able to accomplish so much."
Frenchman Oresme went to the University of Paris in the 1340s, studying theology and liberal arts. Later he was a faculty member and administrator at the same university. Oresme served royalty as an educator and a scholar and, therefore, had support for his research. He translated Aristotle into French. Eventually he was made a bishop. He began to think about mathematics and in particular rates of change, like velocity and acceleration. His work entitled De configurationibus (1350s) contained results in geometry and was the first to present graphs of velocities. The argument we use to show the divergence of the harmonic series was devised by Oresme in this publication. Oresme was a popularizer of science and did not believe in Albert of Saxony's generally accepted model of free fall. Oresme preferred Aristotle's constant-acceleration model, which became popular among Oxford scholars in the 1330s and was eventually refined and tested by Galileo three hundred years later. In another publication, Algorimus proportionum (Algorithms of Ratios), Oresme used fractional exponent notation and built an algebra of exponents.

Major theorem: Divergence of the harmonic series

Major publications: De configurationibus, Algorimus proportionum
Pascal, Blaise (1623 - 1662)

Pascal was born in France and was encouraged by his father to study science. He met Fermat and was inspired to work on applied science problems. As early as 1640, he wrote an essay on conic sections and earned praise for his work from Descartes. Even though he suffered from poor health, Pascal designed an "arithmetic machine" to perform computations for tax collecting. He completed the first operating model in 1642 and built fifty more during the next decade. The machine was a small box with eight dials, each geared to a drum that displayed the digits in a register window.

Pascal also contributed to the development of differential calculus. Later, he became interested in the physics of fluids under pressure and other components and concepts of hydrostatics. After poor health adversely affected his pursuits in science, Pascal became interested in games of chance. This led to his study of probability and his contributions to the foundations of the calculus of probability. Poor health and interest in religious affairs kept him from pursuing mathematics full time; however, he still continued to produce important mathematical results in geometry and algebra.

Quotation:

"When we wish to demonstrate a general theorem, we must give the rule as applied to a particular case; but if we wish to demonstrate a particular case, we must begin with the general rule. For we always find the thing obscure which we wish to prove and that clear which we use for the proof."
Poincaré, Jules-Henri (1854 - 1912)

Poincaré studied mathematics at the École polytechnique and the École des mines. He was a student of Hermite. He held the chair of mathematical physics at the University of Sorbonne. His major work in differential equations and dynamical systems contributed to results in several applied fields including mechanics, optics, electricity, thermodynamics, fluids, quantum theory, and relativity. He also developed significant results in theoretical mathematics, number theory, and algebra. For his expertise in this broad spectrum of science and mathematics, he is described as the last universalist who understood and contributed in virtually all parts of mathematics. Poincaré was the first to consider chaotic systems, which waited over 75 years for computer graphics to be understood and developed as a branch of mathematics.
Poisson, Siméon-Denis (1781 - 1840)

French mathematician Poisson attended the École centrale, then later he entered the École polytechnique. He studied with Lagrange and Laplace and did so well that he was made an assistant professor at the École upon his graduation. In 1806 he replaced Fourier as the professor of mathematics. Poisson's early work in mechanics appeared in his first volume of Traité de mécanique in 1811. He applied his mathematics to applications in physics and mechanics, including elasticity and vibrations. Poisson published a second volume of Traité de mécanique in 1833. This was followed by Théorie mathématique de la chaleur, published in 1835.

Much of his original work was done in the solution and analysis of differential equations. Poisson became one of France's leading mathematicians and a strong spokesman for the subject. Poisson often declared, "Life is good for only two things: to study mathematics and to teach it." At the end of his career, Poisson worked in other subjects and published Recherchés sur la probabilité in 1837, applying calculus and probability to the humanistic sciences, and Recherchés sur le mouvement des projectiles dans l'air in 1839. Poisson declared support for Leibniz over Fermat as the creator of calculus.

Major publications: Traité de mécanique (1811, 1833); Recherchés sur la probabilité; Recherchés sur le mouvement des projectiles dans l'air
Riemann, Bernhard (1826 - 1866)

Riemann was born in Hanover, Germany. He was a brilliant child and read the works of Euler and Legendre as a young boy. He attended the University of Göttingen and the University of Berlin. His doctorate was obtained under the direction of Gauss in the theory of complex variables. He also worked with physicist Wilhelm Weber. During his early work he managed to introduce the foundational ideas of differential geometry. He went on to study and contribute in dynamics, non-Euclidean geometry, and computational physics. Riemann continued to work as a lecturer at Göttingen and eventually became a professor in 1859. Unfortunately, Riemann suffered from tuberculosis and could only work a few hours each day for the last years of his life.
Rolle, Michel (1652 - 1719)

French mathematician Michel Rolle was largely self-educated in mathematics. He worked as an accountant and studied algebra and the Diophantine equations whenever he found time. After Rolle produced an elegant solution to a recreational mathematics problem posed by Ozanam, Rolle received support for his mathematical research and positions in government service. In 1690 he published *Traité d’algèbre*, which contained advances in notation and methods for solving for the roots of equations. The next year, his follow-on work *Démonstration d’une méthode pour resoudre les egalitez de tous les degrez*, contained the theorem that bears his name. When Rolle published his famous theorem in 1691, his goal was to show that between every two zeros of a polynomial function there always lies a zero of the polynomial's derivative. However, Rolle distrusted the new methods of calculus and spent considerable time and energy denouncing its use and attacking L'Hospital's calculus book. It is ironic that Rolle is known today for his contribution to a field he tried to suppress.

Major theorem: Rolle's theorem (1691)

Major publications: *Traité d’algèbre* (1690); *Démonstration d’une méthode pour resoudre les egalitez de tous les degrez* (1691)
Runge, Carl (1856 - 1927)

German mathematical physicist Carl Runge developed numerical methods for solving the differential equations that arose in his studies of atomic spectra. These numerical methods are still used today. He used so much mathematics in his research that physicists thought he was a mathematician, and he did so much physics that mathematicians thought he was a physicist. Runge studied at the University of Munich and later transferred to the University of Berlin to become one of Weierstrass' students. Later he worked with Kronecker on problems in algebra. Runge taught at Technische Hochschule in Hannover for many years and applied mathematics to problems in physics and chemistry.

In 1904, Felix Klein persuaded his Göttingen colleagues to create for Runge Germany’s only full professorship in applied mathematics. Runge was the first and only occupant of this chair. Later Runge helped Klein change the mathematics curricula in Germany to include applications. Today his name is associated with the Runge-Kutta methods to numerically solve differential equations. M. W. Kutta (1867 - 1944), another German applied mathematician, is also remembered for his contribution to the Kutta-Joukowski theory of airfoil lift in aerodynamics.
Simpson, Thomas (1720 - 1761)

Simpson was a successful text writer and did most of his work in probability. He taught at the Royal Military Academy in Woolwich. His first articles were published in the *Ladies’ Diary*. Later he became editor of this popular journal. Simpson's rule to approximate definite integrals was developed and used before he was born. It is another of history's beautiful quirks that one of the ablest mathematicians of the 18th century is remembered not for his own work or his textbooks but for a rule that was never his, that he never claimed, and that bears his name only because he happened to mention it in one of his books.

Major Publications: *A New Treatise of Fluxions*, *The Laws of Chance*  
(Textbooks:) *Algebra, Geometry, Trigonometry*
Snell, Willebrord (1580 - 1626)

Snell was born in Leiden, Holland. His father was the professor of mathematics at the University of Leiden, and Snell studied there from his father. When his father died, Snell succeeded him as professor. Snell traveled around Europe meeting with scientists like Brahe and Kepler. Much of his work applied mathematics to the determination of the size and shape of the earth and to mapmaking and surveying. In 1624 he published *Tiphys batavus*, a work on navigation. In applying mathematics to astronomy, Snell published his findings in two books: *Cyclometricus de circuli dimensione* and *Concerning the Comet*. Snell developed an important result involving the measure of light refraction as it travels into different media. While he never published the result, Descartes did so ten years after Snell's death, and today it is known as Snell's law.

Major publications: *Tiphys batavus; Cyclometricus de circuli dimensione; Concerning the Comet*
Taylor, Brook (1685 - 1731)

Taylor was an ingenious and productive British mathematician. He attended St. John's College in Cambridge, and shortly after his graduation was elected a Fellow of the Royal Society. Taylor published his book on calculus, *Methodus incrementorum directa et inversa* in 1715, and his book on geometry, *Linear Perspective*, in the same year. Brook Taylor did not invent Taylor series, and Maclaurin series were not developed by Maclaurin. James Gregory was working with Taylor series when Taylor was only a few years old. Gregory also published the Maclaurin series for many trigonometric functions ten years before Maclaurin was born.

Taylor was not aware of Gregory's work when he published his book *Methodus incrementorum directa et inversa*, which contained what we now call Taylor series. Maclaurin quoted Taylor's work in a calculus book he wrote in 1742. Maclaurin's book popularized series representations of functions, and although Maclaurin never claimed to have discovered them, Taylor series centered at $a = 0$ later became known as Maclaurin series. History balanced things in the end. Maclaurin, a brilliant mathematician, was the original discoverer of the rule for solving systems of equations that we now call Cramer's rule. Johann Bernoulli also made an independent discovery of Taylor's theorem. Lagrange later considered Taylor series so important that he proclaimed the theorem to be "the fundamental principle of differential calculus."

Major theorem: Taylor's theorem on series

Major publications: *Methodus incrementorum directa et inversa*, *Linear Perspective*
Torricelli, Evangelista (1608 - 1647)

Italian Torricelli attended the University of Rome. He served as Galileo's secretary and succeeded Galileo as a mathematician for the Grand Duke of Tuscany. He applied mathematics to fluid flow and projectile motion. He developed calculus-like procedures for calculating arc length and finding infinitesimals. He also made telescopes and microscopes by designing and grinding fine lenses.
St. Vincent, Gregory (1584 - 1667)

Born in Belgium, St. Vincent studied mathematics at Douai. He was received into the Jesuit order in 1607. In 1613 he was ordained a priest. He taught mathematics at the Jesuit college in Antwerp. There he published Theses de cometis and Theses mechanicae. St. Vincent went to Louvain to teach and work. He asked church officials in Rome for permission to publish his work on quadrature of the circle, but was denied that permission. He was called to Rome to modify his manuscript; however, he was stricken by a stroke before the matter was settled. St. Vincent fled to Ghent to escape the war in Europe, leaving his writings behind. Fortunately his papers were saved and returned to him ten years later in 1641. Finally, his work, completed many years earlier, was published in Antwerp in 1647 as Opus geometricum quadraturae circuli et sectionum coni. This book contained geometric series, conics, quadrature methods (similar to Cavalieri's method), and quadrature of the circle (the error in the integration used was detected by Huygens in 1651). St. Vincent made important contributions to the development of calculus, and his books were read by the next generation of mathematicians as they connected ideas and refined the concepts of calculus.

Major publication: Opus geometricum quadraturae circuli et sectionum coni
Weierstrass, Karl (1815 - 1897)

Weierstrass attended the University of Bonn to learn public administration, but he found that his passion was for mathematics. He read Laplace, Legendre, Jacobi, and Abel. He taught for 14 years at the secondary school level before he published 2 brilliant papers and received an offer to teach at the university level. He became a professor of mathematics at the University of Berlin. In his work to put mathematical analysis on a sound logical foundation, Weierstrass developed the modern definitions of limit (delta-epsilon) and continuity.

In his Berlin lectures in the 1860s, he also proved several theorems for continuous and complex functions. In the first half of the nineteenth century, most mathematicians believed and many texts "proved" that continuous functions were differentiable. Their idea of function was limited to functions defined by algebraic formulas. Weierstrass found counter-examples and opened a whole new thinking about properties of functions. The standards of rigor that he set greatly affected the future of mathematics. Because of severe illness, Weierstrass had an advanced student write on the chalkboard while he lectured. His illness did not deter his enthusiasm for mathematics.