Do as much as you can. Write your answers to all of the questions. Grades greater than or equal to 100 will be 100.

1. Find the area of the region in the first quadrant bounded on the left by the $y$-axis and on the right by the curves $y = \sin x$ and $y = \cos x$. Sketch the region first and then find the area using definite integral.
Let $a < b$ be real numbers. Let $f : [a, b] \rightarrow \mathbb{R}$ be a **continuous** function.

Let $I$ be an **open interval** in $\mathbb{R}$. Let $g_1 : I \rightarrow \mathbb{R}$ and $g_2 : I \rightarrow \mathbb{R}$ be **differentiable** functions such that

$$g_1(x) \in (a, b) \quad \text{and} \quad g_2(x) \in (a, b) \quad \text{for all } x \in I.$$  

Consider the function $h : I \rightarrow \mathbb{R}$ defined for all $x \in I$ by

$$h(x) = \int_{g_1(x)}^{g_2(x)} f(t) \, dt.$$  

Prove that for all $x \in I$,

$$h'(x) = f(g_2(x))g_2'(x) - f(g_1(x))g_1'(x).$$  

**Hint:** Use the **Fundamental Theorem of Calculus**: the function $F(x) = \int_{a}^{x} f(t) \, dt$ is an antiderivative of $f(x)$ on the interval $(a, b)$. Express $h(x)$ in terms of $f(x)$ and differentiate using the chain rule for derivative.

(b) Use part (a) to find for all $x > 0$,

$$\frac{d}{dx} \left[ \int_{\sqrt{x}}^{2\sqrt{x}} \sin(t^2) \, dt \right].$$
3. Evaluate the following indefinite integrals:

(a) \( \int \cos(\ln x) \, dx \).

\textbf{Hint:} Firstly use a \textit{substitution} and then \textit{integration by parts}.
(b) \[ \int \frac{x^3}{\sqrt{x^4 + 4}} \, dx. \]

Hint: Use a trigonometric substitution.
4. Use the method of partial fractions for the integration of rational functions to find the integral
\[ \int \frac{x^4 + 3x^2 + 2x - 2}{x^5 - x^4 + 2x^3 - 2x^2 + x - 1} \, dx. \]

Hint: After the partial fraction decomposition, the reduction formula in the next problem can also be helpful for some of the integrals.
5. Prove the following reduction formula for all integers $n \geq 2$:

$$\int \frac{1}{(1 + x^2)^n} \, dx = \frac{1}{2n - 2} \frac{x}{(1 + x^2)^n - 1} + \frac{2n - 3}{2n - 2} \int \frac{1}{(1 + x^2)^{n-1}} \, dx.$$

**Hint:** Start with \( \frac{1}{(1 + x^2)^n} = \frac{(1 + x^2) - x^2}{(1 + x^2)^n} = \frac{1}{(1 + x^2)^{n-1}} - \frac{x^2}{(1 + x^2)^n} \) and now use integration by parts for the integral \( \int \frac{x^2}{(1 + x^2)^n} \, dx. \)
6. Determine the convergence or divergence of the following improper integrals:

(a) \[ \int_1^\infty \frac{1}{x (\ln x)^2} \, dx \]

(b) \[ \int_0^1 \frac{1}{x \ln x} \, dx \]
7. Find the **volume** of the solid generated by revolving the following region $D$ in the $xy$-plane around the $x$-axis:

$$R : \quad \text{the region in the first quadrant bounded by the curve } x = y - y^3 \text{ and the } y\text{-axis.}$$

Firstly sketch the region $R$ and determine which of the methods (**disk method**, **washer method** or **cylindrical shell method**) could you better use to find the volume of the solid.
8. (a) If the function \( r = f(\theta) \) has a continuous first derivative for \( \alpha \leq \theta \leq \beta \) (where \( \alpha \leq \beta \) in \( \mathbb{R} \)) and if the point

\[
P(r, \theta) = (x, y) = (r \cos \theta, r \sin \theta) = (f(\theta) \cos \theta, f(\theta) \sin \theta)
\]

traces the polar curve \( r = f(\theta) \) exactly once as \( \theta \) runs from \( \alpha \) to \( \beta \), then prove that the length of the curve is

\[
L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left( \frac{dr}{d\theta} \right)^2} \, d\theta.
\]

**Hint:** Use the formula for the length of a parametrized curve.
(b) Sketch the polar curve (a logarithmic spiral)

\[ r = \frac{e^{\theta}}{\sqrt{2}}, \quad 0 \leq \theta \leq \pi \]

and find its length by using the formula in part (a).