EXACTNESS OF ČECH HOMOLOGY THEORY

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Abstract

It is known that the Čech homology sequence of a compact pair \((X, A)\) over a compact coefficient group is exact. It is proved that the Čech homology sequence of a compact pair \((X, A)\) over an algebraically compact coefficient group is exact, too. To show this it is proved that if the Čech homology sequence of a pair \((X, A)\) (not necessarily a compact pair) over a coefficient group \(G\) is exact and \(H\) is a direct summand of the group \(G\), then the Čech homology sequence of the pair \((X, A)\) over the group \(H\) is exact, too. In showing this we work in the category \(\text{Inv}_M(\text{Comp})\) of inverse systems of chain complexes over a fixed directed set \(M\) which is shown to form a category.

Keywords: Čech homology, exactness, algebraically compact.

1 Introduction

To define the Čech homology theory, one uses the formal homology theory of simplicial complexes and inverse limit of inverse systems of groups. Formal homology theory of simplicial complexes is obtained using the homology theory of chain complexes with a coefficient group (coefficient group is introduced using tensor product). In the definition of homology theory of chain complexes and simplicial complexes, the clear language of \(\mathcal{C}\)-categories, \(\partial\)-functors, \(h\)-categories and \(h\)-functors is useful. We turn back to the definitions of the homology theory of simplicial complexes and chain complexes because in the proofs we start from these definitions and use basic properties of tensor product since coefficient groups enter in homology theory by tensor products. To fix these definitions and the definition of the Čech homology theory as well as the terminology, notation and other conventions, we refer to the classical book [1]. For the definition of the Čech homology theory, its further properties and its relation, comparison with other homology theories, we refer also to [2], [3], [4].

Čech homology lacks the exactness axiom required in the Eilenberg-Steenrod axioms for homology theory, but it is useful for some other reasons (see [1, 2, 3, 4]). It forms a distinguished example of so called a partially exact homology theory.

The Čech homology sequence of a pair \((X, A)\) over a coefficient group \(G\) is known to be exact under some restrictions on the pair \((X, A)\) and on the group \(G\). One such is that the pair \((X, A)\) be a compact pair and the group \(G\) be a compact abelian group. In this article, we show that if the pair \((X, A)\) is a compact pair and the group \(G\) is an algebraically compact group, i.e. algebraically a direct summand of a compact group, then exactness is again obtained as proved in Section 4. In showing this, we prove that if the Čech homology sequence of a pair \((X, A)\) over a group \(G\) is exact and \(H\) is a direct summand of the group \(G\), i.e. \(G = H \oplus H'\) for some subgroup \(H'\) of \(G\), then the Čech homology sequence of the pair \((X, A)\) over the group \(H\) is exact, too. To obtain this result we work in the category \(\text{Inv}_M(\text{Comp})\) of inverse systems of chain complexes over a fixed directed set \(M\); this category is described in Section 3.

For an example of a compact abelian group we firstly remind the circle group \(S^1\) of the unit circle in the complex plane \(\mathbb{C}\); its topology is the topology that it inherets as a subspace of \(\mathbb{C}\) and its operation is the ordinary multiplication of complex numbers. This group \(S^1\) is isomorphic to the additive quotient group \(\mathbb{R}/\mathbb{Z}\) of real numbers modulo 1 (\(\mathbb{R}\) denotes the real numbers). Note that we are dealing with abelian groups, so we do not consider nonabelian compact matrix groups, like orthogonal matrices. The method of describing the algebraic structure of a compact abelian group led to the discovery of algebraically compact groups (see [5, §17]). The following are all algebraically compact groups: divisible groups, like the additive group \(\mathbb{Q}\) of rational numbers, the Prüfer group \(\mathbb{Z}(p^\infty)\), \(p\) prime number, and direct sums of such groups; every bounded group (i.e. a group \(A\) such that \(nA = 0\) for some positive integer \(n\)); finite groups; every direct summand of an algebraically compact group; a direct product of algebraically compact groups; the additive group of \(p\)-adic integers; a direct product of copies of the additive group of \(p\)-adic numbers; linearly compact abelian groups; \(\text{Hom}(A, C)\) where \(A\) is an

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arbitrary group and $C$ is an algebraically compact group; $\text{Hom}(A,C)$ where $A$ is a torsion group and $C$ is an arbitrary group. But the additive group $\mathbb{Z}$ of integers is not algebraically compact; more generally nonzero free groups are never algebraically compact. For the equivalent definitions of algebraic compactness, about the examples given, some structure theorems, cardinal invariants for algebraically compact groups, we refer to [6, Ch. VII].

In Section 2, we give the notation we use which may not be standard and give the definition-notation of some basic terms. Section 3 forms the algebraic part dealing with limits of inverse systems of exact sequences used in the proof of the main theorem (Theorem 4.2) of this article at Section 4. The main lemma used in the proof of Theorem 4.2 is proved in Section 4. Exactness of the Čech homology sequence of a pair $(X,A)$ over an algebraically compact group is then just a corollary of Theorem 4.2 as shown in Section 4.

2 Notation and terminology

For any undefined terms or notation, we refer to [1].

By a group, we will always mean an abelian group, although we usually emphasize that the groups are abelian. We use the following notation for the categories we use:

$\mathcal{A} = \text{(the category of abelian groups and their group homomorphisms)}$,

$\mathcal{B}_C = \text{(the category of compact abelian groups and their continuous homomorphisms)}$,

$\text{Comp} = \text{(the category of chain complexes of abelian groups and their chain homomorphisms)}$,

$\mathcal{K}_s = \text{(the category of simplicial complexes and their simplicial maps)}$.

For a homomorphism $f : A \longrightarrow B$ between groups, Ker$(f)$ denotes the kernel of $f$ and Im$(f)$ denotes the image of $f$.

$\mathbb{Z}$ denotes the set of all integers; by a sequence we will usually mean one indexed by $\mathbb{Z}$.

**Definition 2.1.** ([1, §I.2]) A lower sequence $G$ of groups is a collection $\{G_q, \phi_q\}_{q \in \mathbb{Z}}$ or shortly $\{G_q, \phi_q\}$ where for each integer $q$ (positive, negative or zero), $G_q$ is a group, and $\phi_q : G_q \longrightarrow G_{q-1}$ is a homomorphism:

$$G : \quad \ldots \longrightarrow G_{q+1} \xrightarrow{\phi_{q+1}} G_q \xrightarrow{\phi_q} G_{q-1} \longrightarrow \ldots$$

It is said to be exact if Ker$(\phi_q) = \text{Im}(\phi_{q+1})$ for all $q \in \mathbb{Z}$. A lower sequence $G' = \{G'_q, \phi'_q\}$ is said to be a subsequence of the lower sequence $G = \{G_q, \phi_q\}$ if for each $q$, $G'_q \subset G_q$ and $\phi'_q = \phi_q|_{G'_q}$. A subsequence is determined by any set of subgroups $\{G'_q\}$ provided $\phi_q(G'_q) \subset G'_{q-1}$ for each $q$. If $G = \{G_q, \phi_q\}$, $G' = \{G'_q, \phi'_q\}$ are two lower sequences, a chain homomorphism or simply homomorphism $\psi : G \longrightarrow G'$ is a sequence $\{\psi_q\}_{q \in \mathbb{Z}}$ such that, for each integer $q$, $\psi_q : G_q \longrightarrow G'_q$ is a homomorphism and the following commutativity relations hold:

$$\phi'_q \psi_q = \psi_{q-1} \phi_q,$$

that is, the diagram

$$\begin{array}{ccc}
G_{q-1} & \xrightarrow{\phi_q} & G_q \\
\downarrow{\psi_{q-1}} & & \downarrow{\psi_q} \\
G'_{q-1} & \xrightarrow{\phi'_q} & G'_q
\end{array}$$

is commutative.

Denote a chain homomorphism by

$$G : \quad \ldots \longrightarrow G_q \xrightarrow{\phi_q} G_{q-1} \longrightarrow \ldots$$

$$\psi \downarrow \quad \psi_q \downarrow \quad \psi_{q-1} \downarrow$$

$$G' : \quad \ldots \longrightarrow G'_q \xrightarrow{\phi'_q} G'_{q-1} \longrightarrow \ldots$$

The subgroups $\{\text{Ker} \psi_q\}$ form a subsequence of $G$ called the kernel of $\psi$, and $\text{Ker} \psi = 0$ means $\text{Ker} \psi_q = 0$ for each $q$. Likewise $\text{Im} \psi = \{\text{Im} \psi_q\}$ is a subsequence of $G'$; and when $G' = \text{Im} \psi$, we say that $\psi$ is onto. If each $\psi_q$ an isomorphism, then $\psi$ is said to be an isomorphism.
For simplicity, we shall say that a group \( H \) is a \textit{direct summand} of a group \( G \) if there exist subgroups \( H' \) and \( A \) of \( G \) such that \( G = H' \oplus A \) (internal direct sum) and \( H \cong H' \) where \( \cong \) means isomorphic as groups, that is, for some monomorphism \( f : H \longrightarrow G \), the short exact sequence

\[
0 \longrightarrow H \xrightarrow{f} G \xrightarrow{\sigma} G/f(H) \longrightarrow 0
\]

is splitting, where \( \sigma : G \longrightarrow G/f(H) \) is the canonical epimorphism; in that case we can identify \( H \) and \( H' = f(H) \), so consider \( H \) as a subgroup of \( G \).

**Definition 2.2.** Let \( L \) be a subsequence of a lower sequence \( G = \{G_q, \phi_q\} \). \( L \) is said to be a \textit{direct summand} of \( G \) if there exists a subsequence \( L' \) of \( G \) such that for each \( q \in \mathbb{Z} \), \( G_q = L_q \oplus L'_q \). We will say that a lower sequence \( L \) is a \textit{direct summand} of a lower sequence \( G \), if \( G \) has a subsequence \( L \) which is a direct summand of \( G \) in the sense just defined and which is isomorphic to \( L \).

**Proposition 2.3.** Let \( G' = \{G'_q, \phi'_q\} \) and \( G = \{G_q, \phi_q\} \) be lower sequences and \( f : G' \longrightarrow G \) and \( g : G \longrightarrow G' \) be chain homomorphisms such that \( g \circ f = i_{G'} \), where \( i_{G'} : G' \longrightarrow G' \) is the identity chain homomorphism. Then \( G' \) is a direct summand of \( G \) and so \( G'_q \) is a direct summand of \( G_q \) for each \( q \in \mathbb{Z} \).

Proof. \( g \circ f = i_{G'} \) implies \( g_q \circ f_q = i_{G'_q} \) for each \( q \in \mathbb{Z} \), where \( i_{G'_q} : G'_q \longrightarrow G'_q \) is the identity map of \( G'_q \). Then \( G_q = \text{Im}(f_q) \oplus \text{Ker}(g_q) \) for each \( q \in \mathbb{Z} \). \( \text{Im}(f) = \{\text{Im}(f_q)\} \) is a subsequence of \( G \) and \( \text{Ker}(g) = \{\text{Ker}(g_q)\} \) is a subsequence of \( G \), so by Definition 2.2, \( \text{Im}(f) \) is a direct summand of \( G \). Since \( f_q \) is monomorphism for each \( q \), \( \text{Im}(f_q) \cong G'_q \) for each \( q \in \mathbb{Z} \). Thus \( G' \) is isomorphic to the subsequence \( \text{Im}(f) = \{\text{Im}(f_q)\} \) of \( G \) (via \( f \)) which implies that \( G' \) is a direct summand of \( G \) by Definition 2.2.

Categorical language will be used generally in the following sense: By a map of one object into another object in the category we are working in, we mean a map belonging to that category, i.e. a morphism in this category. For example by a map \( f : A \longrightarrow B \) in the category \( \mathcal{A} \) of abelian groups, we mean a group homomorphism. By a map \( f : X \longrightarrow Y \) in the category of topological spaces, we mean a continuous function. By a map \( f : A \longrightarrow B \) in the category \( \mathcal{A}_C \) of compact abelian groups, we mean a group homomorphism which is also a continuous function, i.e. a continuous homomorphism. By a map \( f : A \longrightarrow B \) in the category \( \text{Comp} \) of chain complexes of abelian groups, we mean a chain homomorphism from the chain complex \( A \) to the chain complex \( B \).

**Definition 2.4.** We will use a special diagram pattern in the following sections repeatedly and say that a diagram of the form

\[
\begin{array}{c}
A_2 \\ A_2' \downarrow f_2 \\
A_2 \end{array} \quad \begin{array}{c}
A_1 \\ A_1' \downarrow f_1 \\
A_1 \end{array}
\]

is commutative to mean that the diagram obtained by taking \( f_1 \), \( f_2 \) and leaving out \( g_1 \) and \( g_2 \) is commutative, and the diagram obtained by taking \( g_1 \), \( g_2 \) and leaving out \( f_1 \) and \( f_2 \) is commutative, i.e. the following diagrams

\[
\begin{array}{c}
A_2 \\ A_2' \downarrow f_2 \\
A_2 \end{array} \quad \begin{array}{c}
A_1 \\ A_1' \downarrow f_1 \\
A_1 \end{array}
\]

are commutative, so that we do not draw this pair of diagrams repeatedly.

**Definition 2.5.** ([1, Def. V.2.1]) A \textit{chain complex} \( K \) is a lower sequence \( \{C_q(K), \partial_q\} \) of groups and homomorphisms

\[
\partial_q : C_q(K) \longrightarrow C_{q-1}(K)
\]

such that \( \partial_{q-1} \partial_q = 0 \) for each integer \( q \). \( C_q(K) \) is called the group of \( q \)-chains of \( K \), and \( \partial_q \) is called the boundary homomorphism. A map \( f : K \longrightarrow K' \) of one chain complex into another is a chain homomorphism of lower sequences as defined in Definition 2.1, that is, it is a sequence of homomorphisms \( f_q : C_q(K) \longrightarrow C_q(K') \) defined for each integer \( q \) such that \( f_{q-1} \partial_q = \partial'_q f_q \).
The notation for tensor products, homology theory with a coefficient group on chain complexes and homology theory of $K_X$ with a coefficient group will be as in [1, Ch. V-VI].

For inverse systems, inverse limits and systems of exact sequences, see [1, §VIII.2,3,5].

**Definition 2.6.** ([1, Def. VIII.5.1]) An inverse system of lower sequences $\{S_\alpha(\alpha \in M); \pi_{\alpha}^\beta\}$ (shortly $\{S, \pi\}$) over a directed set $M$ is a function which attaches to each $\alpha \in M$ a lower sequence

$$S_\alpha = \{S_{\alpha,q}, \phi_{\alpha,q}\}_{q \in \mathbb{Z}} : \cdots \xrightarrow{\phi_{\alpha,q}} S_{\alpha,q} \xrightarrow{\phi_{\alpha,q-1}} \cdots$$

and to each relation $\alpha \leq \beta$ in $M$, a homomorphism $\pi_{\alpha}^\beta : S_\beta \longrightarrow S_\alpha$ of lower sequences

$$\begin{align*}
S_\beta : & \quad \cdots \longrightarrow S_{\beta,q} \xrightarrow{\phi_{\beta,q}} S_{\beta,q-1} \longrightarrow \cdots \\
\pi_{\alpha}^\beta & \quad \xrightarrow{\phi_{\beta,q}} \xrightarrow{\phi_{\alpha,q}} \xrightarrow{\phi_{\alpha,q-1}} \cdots \\
S_\alpha : & \quad \cdots \longrightarrow S_{\alpha,q} \xrightarrow{\phi_{\alpha,q}} \cdots
\end{align*}$$

i.e. homomorphisms $\pi_{\alpha,q}^\beta : S_{\beta,q} \longrightarrow S_{\alpha,q}$ for each $q \in \mathbb{Z}$ such that

$$\phi_{\alpha,q} \circ \pi_{\alpha,q}^\beta = \pi_{\alpha,q-1} \circ \phi_{\beta,q}$$

satisfying

$$\pi_{\alpha}^\alpha = \text{identity}, \quad \text{and} \quad \pi_{\alpha}^\beta \pi_{\beta}^\gamma = \pi_{\alpha}^\gamma \quad \text{if} \quad \alpha \leq \beta \leq \gamma \quad \text{in} \quad M$$

Then for any fixed $q$, the groups and homomorphisms $\{S_{\alpha,q}(\alpha \in M); \pi_{\alpha,q}^\beta\}$ form an inverse system of groups over $M$; denote its limit group by $S_{\infty,q}$. Again for a fixed $q$, the homomorphisms $\phi_{\alpha,q} : \alpha \in M$, together with identity map of $M$ form a map

$$\Phi_q : \{S_{\alpha,q}(\alpha \in M); \pi_{\alpha,q}^\beta\} \longrightarrow \{S_{\alpha,q-1}(\alpha \in M); \pi_{\alpha,q-1}^\beta\}$$

of inverse systems. Denote the limit of $\Phi_q$ by

$$\phi_{\infty,q} : S_{\infty,q} \longrightarrow S_{\infty,q-1}.$$ 

The lower sequence $S_{\infty} = \{S_{\infty,q}, \phi_{\infty,q}\}_{q \in \mathbb{Z}}$ so obtained is called the inverse limit of the system $\{S_\alpha(\alpha \in M); \pi_{\alpha}^\beta\}$ (shortly $\{S, \pi\}$). In this definition, it is assumed that all groups and homomorphisms in an inverse system of lower sequences belong to the category $\mathcal{A}$ of abelian groups or to the category $\mathcal{A}_C$ of compact abelian groups; then the limit sequence is also of the same type. Notation of this definition will be used when we work with lower sequences of groups.

### 3 The category $\text{Inv}_M(\text{Comp})$ and limit functor $\text{Inv}_M(\text{Comp}) \rightarrow \text{Comp}$

Remember that by $\text{Comp}$ we denoted the category of chain complexes of abelian groups and their chain homomorphisms. Lower sequences of order 2 are chain complexes (see [1, Definition VIII.5.2]); they are the same according to their definitions but the terminology ‘chain complex’ is used generally if from it homology groups are obtained. The lower sequences we will work in the next section when discussing exactness of Čech homology will themselves be homology sequences (of a simplicial complex) and will not be treated as a chain complex, but we will not create a different notation for the lower sequences of order 2; we will again use $\text{Comp}$ to denote the category of lower sequences of order 2 (=chain complexes) and their chain homomorphisms. We will not care much about this now and use chain complex and lower sequence of order 2 interchangeably.

**Definition 3.1.** Fix a directed set $M$. By $\text{Inv}_M(\text{Comp})$ we will denote the ‘category’ of inverse systems of chain complexes over the directed set $M$. The verification that it is a category with the definition of its objects and maps given below is done in the proof of the theorem that follows.
The objects of \( \textbf{Inv}_M(\textbf{Comp}) \) are inverse systems \( \{S_\alpha (\alpha \in M); \pi_\alpha^3\} \) (shortly \( \{S, \pi\} \)) of lower sequences of order 2 (see Definition 2.6). Maps (morphisms) of \( \textbf{Inv}_M(\textbf{Comp}) \) are defined as follows: For \( \{S_\alpha (\alpha \in M); \pi_\alpha^3\} \) (shortly \( \{S, \pi\} \)) and \( \{S'_\alpha (\alpha \in M); \pi'_\alpha^3\} \) (shortly \( \{S', \pi'\} \)), a map

\[
\Psi : \{S, \pi\} \longrightarrow \{S', \pi'\}
\]

consists of chain homomorphisms

\[
\psi_\alpha : S_\alpha \longrightarrow S'_\alpha, \quad \alpha \in M
\]

such that if \( \alpha \leq \beta \) in \( M \), then commutativity holds in the diagram

\[
\begin{array}{ccc}
S_\alpha & \xrightarrow{\pi_\alpha^3} & S_\beta \\
\psi_\alpha \downarrow & & \downarrow \psi_\beta \\
S'_\alpha & \xrightarrow{\pi'_\alpha^3} & S'_\beta
\end{array}
\]

We denote a map in \( \textbf{Inv}_M(\textbf{Comp}) \) by capital greek letters (like \( \Psi \)) and its ‘components’ by small letters (like \( \psi_\alpha \) as above or by \( \Psi_\alpha \) or by \( (\Psi)_\alpha \)). Composition of two maps

\[
\Psi : \{S, \pi\} \longrightarrow \{S', \pi'\}, \quad \Psi' : \{S', \pi'\} \longrightarrow \{S'', \pi''\}
\]

in \( \textbf{Inv}_M(\textbf{Comp}) \) is defined as the map

\[
(\Psi \Psi) : \{S, \pi\} \longrightarrow \{S'', \pi''\}
\]

given by \((\Psi \Psi)_\alpha = \psi'_\alpha \psi_\alpha, \quad \alpha \in M.\)

Identity map of \( \{S, \pi\} \) in \( \textbf{Inv}_M(\textbf{Comp}) \) is the map

\[
i_{\{S, \pi\}} : \{S, \pi\} \longrightarrow \{S, \pi\}
\]

given by \((i_{\{S, \pi\}})_\alpha = i_{S_\alpha}, \quad \alpha \in M\)

where \( i_{S_\alpha} : S_\alpha \longrightarrow S_\alpha \) is the identity chain homomorphism of \( S_\alpha \). The notation of this definition will be used throughout when working in the category \( \textbf{Inv}_M(\textbf{Comp}) \).

**Theorem 3.2.** \( \textbf{Inv}_M(\textbf{Comp}) \) forms a category.

**Proof.** Firstly, let us verify that the composition \( \Psi \Psi \) in the definition is a map in \( \textbf{Inv}_M(\textbf{Comp}) \). Since \( \Psi' \) and \( \Psi \) are maps in \( \textbf{Inv}_M(\textbf{Comp}) \), we have the following commutative diagrams:

\[
\begin{array}{ccc}
S_\alpha & \xrightarrow{\pi_\alpha^3} & S_\beta \\
\psi_\alpha \downarrow & & \downarrow \psi_\beta \\
S'_\alpha & \xrightarrow{\pi'_\alpha^3} & S'_\beta
\end{array}
\quad
\begin{array}{ccc}
S'_\alpha & \xrightarrow{\pi'_\alpha^3} & S'_\beta \\
\psi'_\alpha \downarrow & & \downarrow \psi'_\beta \\
S''_\alpha & \xrightarrow{\pi''_\alpha^3} & S''_\beta
\end{array}
\]

Composition of chain homomorphisms is a chain homomorphism (in the category \( \textbf{Comp} \)), so

\[
\psi'_\alpha \psi_\alpha : S_\alpha \longrightarrow S''_\alpha
\]

is a chain homomorphism for every \( \alpha \in M \). The diagram

\[
\begin{array}{ccc}
S_\alpha & \xrightarrow{\pi_\alpha^3} & S_\beta \\
\psi'_\alpha \psi_\alpha \downarrow & & \downarrow \psi'_\beta \psi_\beta \\
S''_\alpha & \xrightarrow{\pi''_\alpha^3} & S''_\beta
\end{array}
\]

is commutative as

\[
(\psi'_\alpha \psi_\alpha) \circ \pi_\alpha^3 = \psi'_\alpha \circ (\psi_\alpha \circ \pi_\alpha^3) = (\psi'_\alpha \circ \pi'_\alpha^3 \circ \psi_\alpha) \circ \pi_\beta = \pi''_\alpha \circ (\psi'_\beta \psi_\beta)
\]
because by commutativity of the two previous diagrams

\[ \psi_{\alpha} \circ \pi_{\alpha}^\beta = \pi'_{\alpha} \circ \psi_{\beta} \quad \text{and} \quad \psi_{\alpha}' \circ \pi'_{\alpha} = \pi_{\alpha}^\beta \circ \psi_{\beta} \]

Composition of maps in \( \text{Inv}_M(\text{Comp}) \) is 'associative' because composition of chain homomorphisms is associative: for maps

\[ \Psi : \{S, \pi\} \rightarrow \{S', \pi'\}, \Psi' : \{S', \pi''\} \rightarrow \{S'', \pi'''\}, \Psi'' : \{S'', \pi'''\} \rightarrow \{S''', \pi''''\} \]

in \( \text{Inv}_M(\text{Comp}) \), we have for every \( \alpha \in M, \)

\[ [\Psi' (\Psi') \alpha] = \Psi'' (\psi_{\alpha} \psi_{\alpha}') = (\psi_{\alpha}' \psi_{\alpha}') = [(\Psi'' \Psi') \alpha] \]

which implies \( \Psi'' (\Psi') = (\Psi'' \Psi') \Psi \). For \( \{S, \pi\} \) in \( \text{Inv}_M(\text{Comp}) \), the identity 'map'

\[ \iota_{\{S, \pi\}} : \{S, \pi\} \rightarrow \{S, \pi\} \]

is really a map in \( \text{Inv}_M(\text{Comp}) \), because the diagram

\[
\begin{array}{ccc}
S_{\alpha} & \xrightarrow{\pi_{\alpha}^\beta} & S_{\beta} \\
\downarrow{\iota_{S_{\alpha}}} & & \downarrow{\iota_{S_{\beta}}} \\
S_{\alpha} & \xrightarrow{\pi_{\alpha}^\beta} & S_{\beta}
\end{array}
\]

is clearly commutative as \( \iota_{S_{\alpha}} \) and \( \iota_{S_{\beta}} \) are identity maps of chain complexes. That the map \( \iota_{\{S, \pi\}} \) is an 'identity map' in the category \( \text{Inv}_M(\text{Comp}) \) follows because for maps

\[ \Psi : \{S, \pi\} \rightarrow \{S', \pi'\}, \quad \Psi' : \{S', \pi''\} \rightarrow \{S, \pi\} \]

in \( \text{Inv}_M(\text{Comp}) \), we have for each \( \alpha \in M, \)

\[ (\Psi \circ \iota_{\{S, \pi\}})_{\alpha} = \psi_{\alpha} \iota_{S_{\alpha}} = \psi_{\alpha} \quad \text{and} \quad (\iota_{\{S, \pi\}} \circ \Psi')_{\alpha} = \iota_{S_{\alpha}} \psi_{\alpha}' = \psi_{\alpha}' \]

which give

\[ \Psi \circ \iota_{\{S, \pi\}} = \Psi \quad \text{and} \quad \iota_{\{S, \pi\}} \circ \Psi' = \Psi'. \]

All these show that \( \text{Inv}_M(\text{Comp}) \) forms a category with the Definition 3.1 \( \square \)

**Definition 3.3.** Let \( \Psi : \{S, \pi\} \rightarrow \{S', \pi'\} \) be a map in \( \text{Inv}_M(\text{Comp}) \). We will follow the notation given in the Definitions 3.1 and 2.6. For every \( q \in \mathbb{Z} \) and \( \alpha \leq \beta \) in \( M \), the diagram

\[
\begin{array}{ccc}
S_{\alpha, q} & \xrightarrow{\pi_{\alpha, q}^\beta} & S_{\beta, q} \\
\downarrow{\psi_{\alpha, q}} & & \downarrow{\psi_{\beta, q}} \\
S'_{\alpha, q} & \xrightarrow{\pi'_{\alpha, q}^\beta} & S'_{\beta, q}
\end{array}
\]

is commutative (because the diagram without the \( q \)'s is commutative by the definition of a map in \( \text{Inv}_M(\text{Comp}) \) so the map

\[ \Psi_q : \{S_{\alpha, q} (\alpha \in M); \pi_{\alpha, q}^\beta \} \rightarrow \{S'_{\alpha, q} (\alpha \in M); \pi'_{\alpha, q}^\beta \} \]

which is defined to consist of the identity map of \( M \) and \( (\Psi_q)_{\alpha} = \psi_{\alpha, q}, \alpha \in M \), is a map of inverse systems of abelian groups, i.e. a map in the category \( \text{Inv}(\mathcal{A}) \). Hence we can pass to limit by [1, Theorem VIII.3.14] to obtain a homomorphism

\[ \psi_{\infty, q} : S_{\infty, q} \rightarrow S'_{\infty, q} \]

of groups for each \( q \in \mathbb{Z} \). The map

\[ \psi_{\infty} : S_{\infty} \rightarrow S'_{\infty} \]

of lower sequences defined by those \( \psi_{\infty, q}, q \in \mathbb{Z} \), is called the inverse limit of the map \( \Psi : \{S, \pi\} \rightarrow \{S', \pi'\} \).
Lemma 3.4. The map $\psi_\infty : S_\infty \longrightarrow S'_\infty$ in the previous definition is a chain homomorphism.

Proof. For each $q \in \mathbb{Z}$, the following diagram in $\textbf{Inv}(\mathcal{A})$ is commutative:

$$
\begin{array}{ccc}
\{S_\alpha,q(\alpha \in M); \pi^\beta_{\alpha,q}\} & \xrightarrow{\psi_q} & \{S'_\alpha,q(\alpha \in M); \pi'^\beta_{\alpha,q}\} \\
\phi_q \downarrow & & \downarrow \phi'_q \\
\{S_\alpha,q-1(\alpha \in M); \pi^\beta_{\alpha,q-1}\} & \xrightarrow{\psi_{q-1}} & \{S'_\alpha,q-1(\alpha \in M); \pi'^\beta_{\alpha,q-1}\}
\end{array}
$$

because for every $\alpha \in M$,

$$
\psi_{q-1} \circ \phi_{\alpha,q} = \phi'_{\alpha,q} \circ \psi_{\alpha,q}
$$

as $\psi_\alpha : S_\alpha \longrightarrow S'_\alpha$ is a chain homomorphism. Thus applying the limit functor $\textbf{Inv}(\mathcal{A}) \longrightarrow \mathcal{A}$ by [1, Theorem VIII.3.14], we get the commutative diagram

$$
\begin{array}{ccc}
S_{\infty,q} & \xrightarrow{\psi_\infty,q} & S'_{\infty,q} \\
\phi_\infty \downarrow & & \downarrow \phi'_\infty \\
S_{\infty,q-1} & \xrightarrow{\psi_{\infty,q-1}} & S'_{\infty,q-1}
\end{array}
$$

that is, for every $q \in \mathbb{Z}$,

$$
\psi_{\infty,q-1} \circ \phi_{\infty,q} = \phi'_{\infty,q} \circ \psi_{\infty,q}.
$$

But this means that $\psi_\infty : S_\infty \longrightarrow S'_\infty$ is a chain homomorphism. \hfill \Box

Theorem 3.5. Let $\textbf{Comp}$ denote the category of chain complexes (=lower sequences of order 2) of abelian groups (i.e. groups in $\mathcal{A}$) and their chain homomorphisms, and $\textbf{Inv}_M(\textbf{Comp})$ denote the category of inverse systems of chain complexes of abelian groups over a directed set $M$. Then the operation of assigning an inverse limit $S_\infty$ to each inverse system $\{S_\alpha(\alpha \in M); \pi^\beta_{\alpha}\}$ (shortly $\{S, \pi\}$) in $\textbf{Inv}_M(\textbf{Comp})$ and an inverse limit map $\psi_\infty : S_\infty \longrightarrow S'_\infty$ in $\textbf{Comp}$ to each map $\Psi : \{S, \pi\} \longrightarrow \{S', \pi\}$ in $\textbf{Inv}_M(\textbf{Comp})$ forms a covariant functor from $\textbf{Inv}_M(\textbf{Comp})$ to $\textbf{Comp}$, which we call as the limit functor $\textbf{Inv}_M(\textbf{Comp}) \longrightarrow \textbf{Comp}$.

Proof. By [1, Theorem VIII.5.3], it follows that the limit sequence $S_\infty$ of an inverse system $\{S, \pi\}$ in $\textbf{Inv}_M(\textbf{Comp})$ is also a chain complex, thus in $\textbf{Comp}$, because a chain complex is just a lower sequence of order 2.

For the identity map $i_{\{S, \pi\}} : \{S, \pi\} \longrightarrow \{S, \pi\}$ of $\{S, \pi\}$ in $\textbf{Inv}_M(\textbf{Comp})$,

$$
(i_{\{S, \pi\}})_\infty,q : S_{\infty,q} \longrightarrow S_{\infty,q}
$$

is the identity map for every $q \in \mathbb{Z}$ because it is obtained by applying the limit functor $\textbf{Inv}(\mathcal{A}) \longrightarrow \mathcal{A}$ (by [1, Theorem VIII.3.14]) to the identity map

$$
(i_{\{S, \pi\}}) : \{S_\alpha,q(\alpha \in M); \pi^\beta_{\alpha,q}\} \longrightarrow \{S_\alpha,q(\alpha \in M); \pi^\beta_{\alpha,q}\}
$$

in $\textbf{Inv}(\mathcal{A})$. Thus

$$
(i_{\{S, \pi\}})_\infty : S_\infty \longrightarrow S_\infty
$$

is the identity map of $S_\infty$ in $\textbf{Comp}$.

For maps

$$
\Psi : \{S, \pi\} \longrightarrow \{S', \pi'\}, \quad \Psi' : \{S', \pi\} \longrightarrow \{S'', \pi''\}
$$

in $\textbf{Inv}_M(\textbf{Comp})$, we must check that

$$
(\Psi' \Psi)_\infty = \Psi'_\infty \Psi_\infty.
$$

Let $q \in \mathbb{Z}$.

$$
(\Psi' \Psi)_\infty,q : S_{\infty} \longrightarrow S''_{\infty}
$$

is the limit of the map

$$
(\Psi' \Psi)_q : \{S_\alpha,q(\alpha \in M); \pi^\beta_{\alpha,q}\} \longrightarrow \{S''_\alpha,q(\alpha \in M); \pi''^\beta_{\alpha,q}\}
$$
in **Inv**(\(\mathcal{A}\)). But that map in **Inv**(\(\mathcal{A}\)) is the composition of the maps

\[
\Psi_q': \{S_{\alpha,q}(\alpha \in M); \pi_{\alpha,q}^\beta\} \longrightarrow \{S_{\alpha,q}'(\alpha \in M); \pi_{\alpha,q}^\beta\}
\]

and

\[
\Psi_q: \{S_{\alpha,q}(\alpha \in M); \pi_{\alpha,q}^\beta\} \longrightarrow \{S_{\alpha,q}'(\alpha \in M); \pi_{\alpha,q}^\beta\}
\]

in **Inv**(\(\mathcal{A}\)). So applying the limit functor **Inv**(\(\mathcal{A}\)) \(\longrightarrow\) \(\mathcal{A}\) (by [1, Theorem VIII.3.14]), we get

\[
(\Psi' \Psi)_q = \Psi'_q \Psi_q
\]

for every \(q \in \mathbb{Z}\), which gives

\[
(\Psi \Psi)_\infty = \Psi_\infty \Psi_\infty.
\]

Thus we get the limit functor **Inv**\(_M\)(**Comp**) \(\longrightarrow\) **Comp**.

**Theorem 3.6.** Let \(\{S_{\alpha}(\alpha \in M); \pi_{\alpha}^\beta\}\) (shortly \(\{S, \pi\}\)) and \(\{S_{\alpha}'(\alpha \in M); \pi_{\alpha}'^\beta\}\) (shortly \(\{S', \pi'\}\)) be inverse systems of exact lower sequences of abelian groups over a directed set \(M\). Assume that we have maps

\[
f: \{S', \pi'\} \longrightarrow \{S, \pi\}
\]

and

\[
g: \{S, \pi\} \longrightarrow \{S', \pi'\}
\]

in the category **Inv**\(_M\)(**Comp**) of inverse systems of chain complexes (of groups in \(\mathcal{A}\)) over the directed set \(M\) such that

\[
g \circ f = i_{\{S', \pi'\}}
\]

where \(i_{\{S', \pi'\}}\) is the identity map of \(\{S', \pi'\}\) in **Inv**\(_M\)(**Comp**) so for every \(\alpha \in M\) the chain homomorphisms

\[
f_\alpha: S_{\alpha}' \longrightarrow S_{\alpha}
\]

and

\[
g_\alpha: S_{\alpha} \longrightarrow S_{\alpha}'
\]

satisfy \(g_\alpha \circ f_\alpha = i_{S_{\alpha}'\alpha}\) so each \(S_{\alpha}'\) is a direct summand of \(S_{\alpha}\), hence each \(S_{\alpha}'\) is a direct summand of \(S_{\alpha,q}\) for each integer \(q\). Then

i. \(S'_\infty\) is a direct summand of \(S_\infty\).

ii. If \(S_\infty\) is exact, then \(S'_\infty\) is also exact.

**Proof.**

i. Applying the limit functor **Inv**\(_M\)(**Comp**) \(\longrightarrow\) **Comp** of the previous Theorem 3.5 to

\[
g \circ f = i_{\{S', \pi'\}}
\]

in **Inv**\(_M\)(**Comp**), we get

\[
g_\infty \circ f_\infty = i_{S'_\infty}.
\]

So by Proposition 2.3, \(S'_\infty\) is a direct summand of \(S_\infty\). Hence, \(S_{\infty,q}'\) is a direct summand of \(S_{\infty,q}\) for each \(q \in \mathbb{Z}\).

ii. Since \(f_\infty: S'_\infty \longrightarrow S_\infty\) and \(g_\infty: S_\infty \longrightarrow S'_\infty\) are chain homomorphisms, we get the following ‘commutative’ diagram for every \(q \in \mathbb{Z}\), that is each square in the following diagram is commutative in the sense described in Definition 2.4:

\[
\begin{array}{ccc}
S'_{\infty,q+1} & \xrightarrow{f_{\infty,q+1}} & S'_{\infty,q} \\
\downarrow & & \downarrow \\
S_{\infty,q+1} & \xrightarrow{f_{\infty,q}} & S_{\infty,q}
\end{array}
\begin{array}{ccc}
S'_{\infty,q} & \xrightarrow{g_{\infty,q}} & S'_{\infty,q-1} \\
\downarrow & & \downarrow \\
S_{\infty,q} & \xrightarrow{g_{\infty,q}} & S_{\infty,q-1}
\end{array}
\]

We already know that \(S'_\infty\) is a chain complex, i.e. image \(\subseteq\) kernel. So to show exactness, we must show for each \(q \in \mathbb{Z}\) that

\[
\text{Ker} \phi_{\infty,q} \subseteq \text{Im} \phi'_{\infty,q+1}.
\]
Take \( x \in \text{Ker} \phi'_{\infty,q} \), so \( \phi'_{\infty,q}(x) = 0 \). Then \( f_{\infty,q-1} \circ \phi'_{\infty,q}(x) = 0 \). By commutativity of the previous diagram, \( f_{\infty,q-1} \circ \phi'_{\infty,q} = \phi_{\infty,q} \circ f_{\infty,q} \). So we get, \( \phi_{\infty,q}(f_{\infty,q}(x)) = 0 \), that is, \( f_{\infty,q}(x) \in \text{Ker} \phi_{\infty,q} \). By exactness of \( S_{\infty} \), \( \text{Ker} \phi_{\infty,q} = \text{Im} \phi_{\infty,q+1} \). Thus, \( f_{\infty,q}(x) = \phi_{\infty,q+1}(y) \) for some \( y \in S_{\infty,q+1} \). Then, \( g_{\infty,q} \circ f_{\infty,q}(x) = g_{\infty,q} \circ \phi_{\infty,q+1}(y) \). By commutativity of the previous diagram, \( g_{\infty,q} \circ \phi_{\infty,q+1} = \phi'_{\infty,q+1} \circ f_{\infty,q+1} \) and since \( g_{\infty,q} \circ f_{\infty,q} = i_{S_{\infty}'} \), \( g_{\infty,q} \circ f_{\infty,q} = i_{S_{\infty}'} \) where \( i_{S_{\infty}'} \) is the identity map of \( S_{\infty}' \). So we obtain \( i_{S_{\infty}'}(x) = \phi'_{\infty,q+1}(f_{\infty,q+1}(y)) \) in \( \text{Im} \phi'_{\infty,q+1} \) as required.

**Corollary 3.7.** If in the previous Theorem 3.6, we assume further that \( \{S, \pi\} \) is an inverse system of exact lower sequences over \( M \) where all groups and homomorphisms of \( \{S, \pi\} \) belong to the category \( \mathcal{A}_C \) of compact abelian groups (so also belongs to the category \( \mathcal{A} \) of abelian groups), then it follows that \( S_{\infty}' \) is exact as well as \( S_{\infty} \) and is a direct summand of \( S_{\infty} \), so \( S_{\infty}' \) is an exact lower sequence of algebraically compact groups and homomorphisms (just ordinary group homomorphisms).

**Proof.** By [1, Theorem VIII.5.6], \( S_{\infty} \) is an exact lower sequence. So by Theorem 3.6, \( S_{\infty}' \) is a direct summand of \( S_{\infty} \) and it is exact. Since for each \( q \in \mathbb{Z} \), \( S_{\infty,q} \) is a direct summand of \( S_{\infty,q} \), which is a compact abelian group, each \( S_{\infty,q} \) is algebraically compact by definition.

### 4 Exactness of the Čech Homology Sequence of a Compact Pair over an Algebraically Compact Coefficient Group

We use the notation of [1, Ch. IX] for the definition of the Čech homology theory. So, for example, all open coverings of a pair \((X, A)\) is denoted by \(\text{Cov}(X, A)\) which is a directed set with respect to the refinement relation; for \(\alpha \in \text{Cov}(X, A)\), \((X_{\alpha}, A_{\alpha})\) denotes the nerve of \(\alpha\), and so on.

The Čech homology sequence of a pair \((X, A)\) over any coefficient group \(G\) is isomorphic with the adjusted homology sequence which is the inverse limit of the inverse system \(\{S_{\alpha}(\alpha \in \text{Cov}(X, A)); \pi_{\alpha}^\beta\} \) (shortly \(\{S, \pi\}\)) of lower sequences where \(S_{\alpha}\) is the homology sequence of the simplicial pair \((X_{\alpha}, A_{\alpha})\) over \(G\) for each \(\alpha \in \text{Cov}(X, A)\), and \(\pi_{\alpha}^\beta : S_{\beta} \longrightarrow S_{\alpha}\) is a chain homomorphism of the homology sequences induced by a projection \((X_{\beta}, A_{\beta}) \longrightarrow (X_{\alpha}, A_{\alpha})\) (see [1, §I.X.7]). With this result, the question of the exactness of the Čech homology sequence of a pair \((X, A)\) is replaced by the question of the exactness of the adjusted sequences. The adjusted sequences are however limits of systems of exact sequences defined over the directed set \(\text{Cov}(X, A)\). Thus the results of Section 3 may be applied.

[1, Theorem VIII.5.3] yields that for any pair \((X, A)\) and any abelian group \(G \in \mathcal{A}\), the Čech homology sequence is a sequence of order 2 (by [1, Theorem IX.7.6]). If \((X, A)\) is a compact pair, then, in defining the groups occuring in the homology sequence, we may limit our attention to finite coverings. If \(G\) is a compact abelian group, then for each finite covering \(\alpha\), the homology sequence of \((X_{\alpha}, A_{\alpha})\) over \(G\) is composed of compact groups and therefore by [1, Theorem VIII.5.6] the limit sequence is exact, that is we have: If \((X, A)\) is a compact pair and \(G\) is a compact abelian group, i.e. \(G \in \mathcal{A}_C\), then the Čech homology sequence of \((X, A)\) over \(G\) is exact (by [1, Theorem IX.7.6]).

But it is not true that the full exactness axiom holds for any group \(G\) even for compact pairs. In [1, §X.4] a compact pair is constructed such that the Čech homology sequence with coefficient group \(\mathbb{Z}\) (the integers) is not exact. But still we can enlarge the class of coefficient groups which produce exact Čech homology sequences for all compact pairs to the class of algebraically compact groups.

**Lemma 4.1.** Let \(H\) be a direct summand of a group \(G\), say

\[ G = H \oplus H' \]

for some group \(H'\). To each simplicial pair \((K, L)\), we associate a pair of chain homomorphisms

\[ f_{K,L} : S'_{K,L} \longrightarrow S_{K,L} \quad \text{and} \quad g_{K,L} : S_{K,L} \longrightarrow S'_{K,L} \]

where
where \( S'_{K,L} \) is the homology sequence of the simplicial pair \((K, L)\) over the coefficient group \(H\) and \( S_{K,L} \) is the homology sequence of the simplicial pair \((K, L)\) over the coefficient group \(G\) (in the formal homology theory of simplicial complexes):

\[
\begin{align*}
S'_{K,L} & \quad \Rightarrow \quad \text{(the homology sequence of \((K, L)\) over the coefficient group \(H\))} \\
S_{K,L} & \quad \Rightarrow \quad \text{(the homology sequence of \((K, L)\) over the coefficient group \(G\))}
\end{align*}
\]

such that

\[
g_{K,L} \circ f_{K,L} = i_{S'_{K,L}}
\]

where \( i_{S'_{K,L}} \) is the identity map of the lower sequence \( S'_{K,L} \).

This assignment of a pair of chain homomorphisms to each simplicial pair is such that: If \( p : (K, L) \rightarrow (K', L') \) is a simplicial map of a simplicial pair \((K, L)\) to a simplicial pair \((K', L')\), then the following diagram is commutative:

\[
\begin{array}{ccc}
S_{K',L'} & \overset{p^*}{\leftarrow} & S_{K,L} \\
\downarrow_{g_{K',L'}} & & \downarrow_{g_{K,L}} \\
S'_{K',L'} & \overset{p'_*}{\leftarrow} & S'_{K,L}
\end{array}
\]

where \( p_* \) and \( p'_* \) denotes the chain homomorphisms of corresponding homology sequences induced by the simplicial map \( p \).

**Proof.** The homology groups \( H_q(K, L; G) \) of a simplicial pair \((K, L)\) over a coefficient group \(G\) is obtained through the following functors (see [1, Ch. IV-V, §VI.1-4]):

\[
(K, L) \xrightarrow{O} K_o/L_o \otimes G(K_o/L_o) \otimes G \xrightarrow{H_q} H_q((K_o/L_o) \otimes G)
\]

For the couple \((i, j)\), where \( i : L \rightarrow K \) and \( j : K \rightarrow (K, L) \) are inclusion maps,

\[
L \xrightarrow{i} K \xrightarrow{j} (K, L)
\]

in the h-category \( K \) of simplicial pairs, applying the covariant h-functor \( O : K \rightarrow \text{Comp} \) gives the direct couple, \((i_o, j_o)\) in \text{Comp}, i.e. the (split) short exact sequence

\[
0 \rightarrow L_o \xrightarrow{i_o} K_o \xrightarrow{j_o} K_o/L_o \rightarrow 0
\]

in \text{Comp}. Since \( \cdot \otimes G \) and \( \cdot \otimes H \) are covariant h-functors from \text{Comp} to \text{Comp}, we get by tensoring with \( H \) and \( G \) the following (split) short exact sequences:

\[
0 \rightarrow L_o \otimes H \xrightarrow{i_o \otimes i_H} K_o \otimes H \xrightarrow{j_o \otimes i_H} (K_o/L_o) \otimes H \rightarrow 0,
\]

\[
0 \rightarrow L_o \otimes G \xrightarrow{i_o \otimes i_G} K_o \otimes G \xrightarrow{j_o \otimes i_G} (K_o/L_o) \otimes G \rightarrow 0
\]

in \text{Comp}, where \( i_H : H \rightarrow H \) and \( i_G : G \rightarrow G \) are identity maps and by the map

\[
i_o \otimes i_H : L_o \otimes H \rightarrow K_o \otimes H
\]
we mean the chain homomorphism defined by
\[(i_o \otimes i_H)_q : C_q(L_o \otimes H) = C_q(L_o) \otimes H \to C_q(K_o \otimes H) = C_q(K_o) \otimes H,\]
\[(i_o \otimes i_H)_q = (i_q)_o \otimes i_H \quad \text{for each } q \in \mathbb{Z},\]
and the maps \(j_o \otimes i_H, i_o \otimes i_G, j_o \otimes i_G\) are similarly defined. We define two maps between direct couples
\((i_o \otimes i_H, j_o \otimes i_H)\) and \((i_o \otimes i_G, j_o \otimes i_G)\) in the category \(\text{Comp}\) as follows:

\[
\begin{array}{ccc}
0 & \to & L_o \otimes H \\
\downarrow f_1 & & \downarrow f_2 \downarrow f_3 \\
0 & \to & L_o \otimes G \\
\end{array}
\]

where we set
\[f_1 = i_{L_o} \otimes in_H, \quad g_1 = i_{L_o} \otimes ph;\]
\[f_2 = i_{K_o} \otimes in_H, \quad g_2 = i_{K_o} \otimes ph;\]
\[f_3 = i_{K_o/L_o} \otimes in_H, \quad g_3 = i_{K_o/L_o} \otimes ph.\]

Here \(in_H : H^c \to G = H \otimes H'\) is the inclusion map of \(H\) in \(G\) and \(ph : G = H \otimes H' \to H\) is the projection of \(G\) onto its direct summand \(H\); \(i_{L_o}\) is the identity map of the chain complex \(L_o\), similarly \(i_{K_o}\) and \(i_{K_o/L_o}\) are identity maps of corresponding chain complexes.

\(f_1, f_2, f_3, g_1, g_2\) and \(g_3\) are chain homomorphisms; let’s verify this for \(f_1\), the other cases are just similar. The map \(f_1 = i_{L_o} \otimes in_H\) consists of the homomorphisms
\[(f_1)q = C_q(L_o \otimes H) = C_q(L_o) \otimes H \to C_q(L_o \otimes G) = C_q(L_o) \otimes G,
(f_1)_q = i_{C_q(L_o)} \otimes in_H, \quad q \in \mathbb{Z},\]
where \(i_{C_q(L_o)} : C_q(L_o) \to C_q(L_o)\) is the identity map of \(C_q(L_o)\). \(f_1\) acts only on the \(H\) part, keeping the other part fixed by the identity map so that as we show it gives a chain homomorphism. Let \(q \in \mathbb{Z}\). Denote by
\[\partial_q : C_q(L_o) \to C_{q-1}(L_o)\]
the boundary homomorphism of the chain complex \(L_o\). We must show commutativity of the square in the diagram

\[
\begin{array}{ccc}
L_o \otimes H : & \cdots & C_q(L_o) \otimes H \\
\downarrow f_i & & \downarrow \partial_q \otimes i_H \\
L_o \otimes G : & \cdots & C_q(L_o) \otimes G \\
\end{array}
\]

\[
\begin{array}{ccc}
& i_{C_q(L_o) \otimes in_H} & \\
\downarrow i_{C_{q-1}(L_o) \otimes in_H} & & \downarrow i_{C_{q-1}(L_o) \otimes in_H} \\
& \partial_q \otimes i_G & \\
\end{array}
\]

to conclude that \(f_1 : L_o \otimes H \to L_o \otimes G\) is a chain homomorphism, that is we must verify the following equality:
\[(\partial_q \otimes i_G) \circ (i_{C_q(L_o) \otimes in_H}) = (i_{C_{q-1}(L_o) \otimes in_H}) \circ (\partial_q \otimes i_H).\]

This equality is obtained by checking it on the generators \(x \otimes h, x \in C_q(L_o), h \in H\) of \(C_q(L_o) \otimes H\) as follows:
\[(\partial_q \otimes i_G) \circ (i_{C_q(L_o) \otimes in_H})(x \otimes h) = (\partial_q \otimes i_G)(x \otimes h) = (\partial_q(x)) \otimes h,
(i_{C_{q-1}(L_o) \otimes in_H}) \circ (\partial_q \otimes i_H)(x \otimes h) = (i_{C_{q-1}(L_o) \otimes in_H})(\partial_q(x)) \otimes h = (\partial_q(x)) \otimes h.\]

This shows that \(f_1\) is a chain homomorphism. Similarly we get that \(f_2, f_3, g_1, g_2\) and \(g_3\) are chain homomorphisms.
To verify that \( f_1, f_2, f_3 \) form a map of the direct couple \((i_o \otimes i_H, j_o \otimes i_H)\) to the direct couple \((j_o \otimes i_G, j_o \otimes i_G)\) and \(g_1, g_2, g_3\) form a map of the couple \((j_o \otimes i_G, j_o \otimes i_G)\) to the couple \((i_o \otimes i_H, j_o \otimes i_H)\), we must show that each square in the diagram is commutative (in the sense described in Definition 2.4). Commutativity of the diagram with only \(f\)'s is obtained from the following equalities:

\[
\begin{align*}
f_2 \circ (i_o \otimes i_H) &= (i_{K_o} \otimes \text{in}_H) \circ (i_o \otimes i_H) = (i_{K_o} \circ i_o) \otimes (\text{in}_H \circ i_H) = i_o \otimes \text{in}_H, \\
(i_o \otimes i_G) \circ f_1 &= (i_o \otimes i_G) \circ (i_{L_o} \otimes \text{in}_H) = (i_o \circ i_{L_o}) \otimes (i_G \circ \text{in}_H) = i_o \otimes \text{in}_H, \\
f_3 \circ (j_o \otimes i_H) &= (i_{K_o}/L_o \otimes \text{in}_H) \circ (j_o \otimes i_H) = (i_{K_o}/L_o \circ j_o) \otimes (\text{in}_H \circ i_H) = j_o \circ \text{in}_H, \\
(j_o \otimes i_G) \circ f_2 &= (j_o \otimes i_G) \circ (i_{K_o} \otimes \text{in}_H) = (j_o \circ i_{K_o}) \otimes (i_G \circ \text{in}_H) = j_o \circ \text{in}_H.
\end{align*}
\]

This shows that \( f_1, f_2, f_3 \) form a map the couple \((i_o \otimes i_H, j_o \otimes i_H)\) to the couple \((j_o \otimes i_G, j_o \otimes i_G)\). Similarly \(g_1, g_2, g_3\) form a map of the couple \((j_o \otimes i_G, j_o \otimes i_G)\) to the couple \((i_o \otimes i_H, j_o \otimes i_H)\).

Let \( q \in Z \). Since \( H_q \) is a covariant \( \theta \)-functor on the \( h \)-category \textbf{Comp} of chain complexes, it must satisfy axiom 3 for the mapping of couples (see [1, §IV.8]); so we have the following commutative diagram:

\[
\begin{array}{ccc}
H_q((K_o/L_o) \otimes H) & \xrightarrow{f_3} & H_q((K_o/L_o) \otimes G) \\
\downarrow & & \downarrow \\
H_{q-1}(L_o \otimes H) & \xrightarrow{f_1} & H_{q-1}(L_o \otimes G)
\end{array}
\]

This is the main part to check in obtaining chain maps \( f_{K,L} \) and \( g_{K,L} \) of the homology sequences:

\[
\begin{align*}
S'_{K,L} : & \cdots \to H_q(L_o \otimes H) \to H_q(K_o \otimes H) \to H_q((K_o/L_o) \otimes H) \xrightarrow{g_3} H_{q-1}(L_o \otimes H) \to \cdots \\
\uparrow f_{K,L} & \downarrow g_{K,L} & \uparrow f_1 & \downarrow g_1 & \uparrow f_2 & \downarrow g_2 & \uparrow f_3 & \downarrow g_3 & \uparrow f_{K,L} & \downarrow g_{K,L}
\end{align*}
\]

\[
\begin{align*}
S_{K,L} : & \cdots \to H_q(L_o \otimes G) \to H_q(K_o \otimes G) \to H_q((K_o/L_o) \otimes G) \xrightarrow{\partial_s} H_{q-1}(L_o \otimes G) \to \cdots \\
\uparrow f_{K,L} & \downarrow g_{K,L} & \uparrow f_1 & \downarrow g_1 & \uparrow f_2 & \downarrow g_2 & \uparrow f_3 & \downarrow g_3 & \uparrow f_{K,L} & \downarrow g_{K,L}
\end{align*}
\]

So we must show commutativity of the three squares in the following diagram:

\[
\begin{array}{ccc}
H_q(L_o \otimes H) & \xrightarrow{(i_o \otimes \text{in}_H)} & H_q(K_o \otimes H) \\
\downarrow f_1 & & \downarrow g_1 \\
H_q(L_o \otimes G) & \xrightarrow{(i_o \otimes \text{in}_G)} & H_q(K_o \otimes G)
\end{array}
\]

\[
\begin{array}{ccc}
H_q((K_o/L_o) \otimes H) & \xrightarrow{(j_o \otimes \text{in}_H)} & H_q((K_o/L_o) \otimes G) \\
\downarrow f_3 & & \downarrow g_3 \\
H_q((K_o/L_o) \otimes G) & \xrightarrow{(j_o \otimes \text{in}_G)} & H_{q-1}(L_o \otimes G)
\end{array}
\]

\[
\begin{array}{ccc}
H_q(L_o \otimes H) & \xrightarrow{\partial_s} & H_{q-1}(L_o \otimes H) \\
\downarrow & & \downarrow \\
H_q(K_o \otimes H) & \xrightarrow{\partial_s} & H_{q-1}(L_o \otimes G)
\end{array}
\]

Commutativity of the first two squares follows from the commutativity of this part when the \( H_q \) functor has not been applied.

So through the above procedure described, we associate to each simplicial pair \((K, L)\) a pair of chain homomorphisms \( f_{K,L} \) and \( g_{K,L} \):
For each $i = 1, 2, 3$, $g_i \circ f_i = \text{id}$ because $p_H \circ i_H = i_H$ where $i_H : H \rightarrow H$ is the identity map of $H$. For example, for $f_1$,

$$g_1 \circ f_1 = (i_{L_0} \otimes p_H) \circ (i_{L_0} \otimes i_H) = (i_{L_0} \circ i_{L_0}) \circ (p_H \circ i_H) = i_{L_0} \otimes i_H = i_{L_0} \otimes H$$

where $i_{L_0} : L_0 \rightarrow L_0 \otimes H$ is the identity map $L_0 \otimes H$.

Since $g_i \circ f_i = \text{id}$, it follows by applying the functor $H_q$ that,

$$g_{i*} \circ f_{i*} = \text{id}.$$  

Thus,

$$g_{K,L} \circ f_{K,L} = \text{id}_{S_{K,L}}.$$  

It remains to verify the commutativity of the following diagram for a simplicial map $p : (K, L) \rightarrow (K', L')$.

![Diagram](image)

We just use a primed notation for the simplicial pair $(K', L')$ in the above procedure of assigning a pair of chain homomorphisms

$$S'_{K', L'} \xleftarrow{g_{K', L'}} S_{K,L},$$

that is in constructing these homomorphisms we use chain complexes $L'_o$, $K'_o$ and maps $f'_i$, $g'_i$ for $i = 1, 2, 3$. To obtain this result, we must show commutativity of the following three diagrams:

![Diagrams](image)

All these three diagrams are commutative since they are obtained by applying the functor $H_q$ to the following three commutative diagrams in $\text{Comp}$:

![Diagrams](image)
Here \( i_H : H \longrightarrow H \) and \( i_G : G \longrightarrow G \) are identity maps, \( \tilde{p} \) and \( \tilde{p} \) are the simplicial maps 
\[
\tilde{p} : L \longrightarrow L', \quad \tilde{p} : K \longrightarrow K',
\]
which are induced by the simplicial map \( p : (K, L) \longrightarrow (K', L') \), \( \tilde{p} \), \( \tilde{p} \) and \( p \) are obtained by applying the functor \( O : K \longrightarrow \text{Comp} \) to \( \tilde{p} \), \( \tilde{p} \) and \( p \).

Commutativity of these three diagrams are easily obtained. Let's check the first one for \( f_1 \):
\[
(\tilde{p} \circ i_G) \circ f_1 = (\tilde{p} \circ i_G) \circ (i_L \circ \text{in}_H) = (\tilde{p} \circ i_G) \circ (i_G \circ \text{in}_H) = \tilde{p} \circ \text{in}_H,
\]
and
\[
f_1 \circ (\tilde{p} \circ i_H) = (i_L' \circ \text{in}_H) \circ (\tilde{p} \circ i_H) = (i_L' \circ \tilde{p}) \circ (\text{in}_H \circ i_H) = \tilde{p} \circ \text{in}_H.
\]

So,
\[
(\tilde{p} \circ i_G) \circ f_1 = f_1' \circ (\tilde{p} \circ i_H)
\]
which means commutativity of the first diagram with \( f_1 \) and \( f_1' \). The other cases are similarly obtained; they hold simply because \( f_i \)'s and \( g_i \)'s act only on the \( H \) and \( G \) parts and \( p \) does not effect these parts.

This ends the proof of the lemma.

\[\square\]

**Theorem 4.2.** Let \( (X, A) \) be a pair of topological spaces and \( G \) be an abelian group such that the Čech homology sequence of \( (X, A) \) over the coefficient group \( G \) is exact. If \( H \) is a direct summand of \( G \), then the Čech homology sequence of \( (X, A) \) over \( H \) is also exact.

**Proof.** Say \( G = H \oplus H' \) for some subgroup \( H' \) of \( G \). Denote by \( M \), the directed set \( \text{Cov}(X, A) \). \( \text{Cov}(X, A) \).

For each \( \alpha \in M \), let
\[
S_\alpha = (\text{the homology sequence of the simplicial pair } (X, A) \text{ over the coefficient group } H),
\]
\[
S_\alpha = (\text{the homology sequence of the simplicial pair } (X, A) \text{ over the coefficient group } G).
\]

If \( \alpha \leq \beta \) in \( M = \text{Cov}(X, A) \), let
\[
p : (X_\beta, A_\beta) \longrightarrow (X_\alpha, A_\alpha)
\]
be a projection. Let
\[
\pi_\alpha^\beta : S_\beta \longrightarrow S_\alpha
\]
be the chain homomorphism from the homology sequence \( S_\beta \) of \( (X_\beta, A_\beta) \) over \( G \) to the homology sequence \( S_\alpha \) of \( (X_\alpha, A_\alpha) \) over \( G \) induced by the projection \( p \). Let
\[
\pi_\alpha^\beta : S_\beta' \longrightarrow S_\alpha'
\]
be the chain homomorphism from the homology sequence \( S_\beta' \) of \( (X_\beta, A_\beta) \) over \( H \) to the homology sequence \( S_\alpha' \) of \( (X_\alpha, A_\alpha) \) over \( H \) induced by the projection \( p \). These give us naturally inverse systems \( \{S_\alpha(\alpha \in M) ; \pi_\alpha^\beta\} \) (shortly \( \{S', \pi\} \)) and \( \{S'_\alpha(\alpha \in M) ; \pi_\alpha^\beta\} \) (shortly \( \{S, \pi\} \)) of lower exact sequences over \( M = \text{Cov}(X, A) \). The Čech homology sequence of \( (X, A) \) is isomorphic to the adjusted homology sequence which is the limit \( S_\infty \) of \( \{S_\alpha(\alpha \in M) ; \pi_\alpha^\beta\} \) and which is assumed to be exact by our hypothesis.

By Lemma 4.1, for each \( \alpha \in M \), we have chain homomorphisms
\[
f_\alpha : S'_\alpha \longrightarrow S_\alpha \quad \text{and} \quad g_\alpha : S_\alpha \longrightarrow S'_\alpha
\]
such that
\[
g_\alpha \circ f_\alpha = i_{S'_\alpha}
\]
where \( i_{S'_\alpha} \) is the identity map of the lower sequence \( S'_\alpha \). To obtain maps
\[
f : \{S', \pi\} \longrightarrow \{S, \pi\} \quad \text{and} \quad g : \{S, \pi\} \longrightarrow \{S', \pi\}
\]
in \( \text{Inv}_M(\text{Comp}) \) using these \( f_\alpha \)'s and \( g_\alpha \)'s, we need to show that for \( \alpha \leq \beta \) in \( M \) the following diagram is commutative:
\[
\begin{array}{ccc}
S_\alpha & \xrightarrow{\pi_\alpha^\beta} & S_\beta \\
\downarrow{g_\alpha} & & \downarrow{f_\beta} \\
S'_\alpha & \xrightarrow{\pi_\alpha^\beta} & S'_\beta
\end{array}
\]
This follows from Lemma 4.1 as $\pi_\alpha^\beta$ and $\pi'_\alpha$ are the chain homomorphisms of corresponding homology sequences induced by the same simplicial map $p: (X_\beta, A_\beta) \longrightarrow (X_\alpha, A_\alpha)$.

Thus we get maps

$$f: \{S', \pi'\} \longrightarrow \{S, \pi\} \quad \text{and} \quad g: \{S, \pi\} \longrightarrow \{S', \pi'\}$$

in the category in $\text{Inv}_M(\text{Comp})$ such that

$$g \circ f = i_{\{S', \pi'\}}$$

as $g_\alpha \circ f_\alpha = i_{S'_\alpha}$ for every $\alpha \in M$. Then by Theorem 3.6 we get our result that $S'_\infty$ is exact as $S_\infty$ is exact.

Since $S'_\infty$ is the adjusted Čech homology sequence of $(X, A)$ over the coefficient group $H$ and since this adjusted sequence is isomorphic to the Čech homology sequence of $(X, A)$ over $H$, we get that the Čech homology sequence of $(X, A)$ over the coefficient group $H$ is exact.

Corollary 4.3. Let $(X, A)$ be a compact pair and $H$ be an algebraically compact group. Then the Čech homology sequence of $(X, A)$ over the coefficient group $H$ is exact.

Proof. Since $H$ is algebraically compact, $H$ is a direct summand of a compact group $G$. Since $G$ is compact, the Čech homology sequence of the compact pair $(X, A)$ over the coefficient group $G$ is exact by [1, Theorem IX.7.6]. By Theorem 4.2, the Čech homology sequence of the compact pair $(X, A)$ over the group $H$, which is a direct summand of $G$, is also exact.

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ČECH HOMOLOJİ TEORİSİİNİN TAMLIĞI

ÖZET

Kompakt bir \((X, A)\) ikilisinin kompakt katsayı gruplu Čech homoloji dizisinin tam olduğu bilinmektedir. Kompakt bir \((X, A)\) ikilisinin cebirsel kompakt katsayı gruplu Čech homoloji dizisinin de tam olduğu ispatlanmaktadır. Bunun için şu gösterilir: Bir \((X, A)\) ikilisin (kompakt bir ikili olması gerekli deildir) bir \(G\) grubu üzerindeki Čech homoloji dizisi tam ise ve \(H\) grubu \(G\) grubunun bir direkt toplam terimi ise, \((X, A)\) ikilisinin \(H\) katsayı gruplu Čech homoloji dizisi de tamdır. Bunu kanıtlarken zincir komplekslerinin sabitlenmiş bir \(M\) yönlendirilmiş kümesi üzerindeki ters sistemlerinden oluşan \(\text{Inv}_M(\text{Comp})\) kategorisinde (bir kategori olduğu gösterilir) çalışılır.

Anahtar Sözcüklar: Čech homoloji, tamlık, cebirsel kompakt.