1) Determine which of the sets is a subspace of \( \mathbb{R}^3 \)

a) \( W_1 = \{ (x_1, x_2, x_3)^T \mid x_1 + x_3 = 1 \} \)

b) \( W_2 = \{ (x_1, x_2, x_3)^T \mid x_3 = x_1 + x_2 \} \)

2) a) Which of the following are linear transformations?

i) \( L_1 : \mathbb{R}^2 \rightarrow \mathbb{R}^4 \) be defined by

\[
L_1((x_1, x_2)^T) = (-x_2, -x_1, x_1, x_2)^T
\]

ii) \( L_2 : \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R} \) be defined by

\[
L_2(A) = \det(A)
\]

b) Find matrix representing \( L_1 \) relative to standard ordered bases for \( \mathbb{R}^2 \) and \( \mathbb{R}^4 \).

3) Let

\[
\mathbf{u}_1 = \left( \frac{1}{3}, \frac{1}{3}, -\frac{4}{3} \right)^T, \quad \mathbf{u}_2 = \left( \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right)^T, \quad \mathbf{u}_3 = (1, -1, 0)^T
\]

Consider the set \( \{ \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \} \)

a) Is the set linearly independent?

b) Does the set form an orthogonal set of vectors?

c) Obtain an orthonormal set from this set.

4) Consider the set \( \{ 1, \cos x, \sin x \} \)

a) Show that this set is an orthogonal set of functions on \( C[-\pi, \pi] \) with respect to inner product defined by

\[
<f, g> = \int_{-\pi}^{\pi} f(x)g(x)dx.
\]

b) Obtain an orthonormal set of functions from this set.

5) Find the eigenvalues and the corresponding eigenspaces for the following matrix

\[
A = \begin{pmatrix}
1 & 2 & 1 \\
0 & 3 & 1 \\
0 & 5 & -1
\end{pmatrix}
\]

6) Let \( L : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) be defined by

\[
L((x_1, x_2)^T) = (x_1 - x_2, x_1, 2x_1 + x_2)^T
\]

a) Prove that \( L \) is a linear transformation

b) Find matrix representing \( L \) relative to standard ordered bases for \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \).
c) Find matrix representing \( L \) relative to ordered bases \( \{(1, 2), (2, 3)\} \) for \( \mathbb{R}^2 \) and \( \{(1, 1, 0), (0, 1, 1), (2, 2, 3)\} \) for \( \mathbb{R}^3 \).

d) Find Kernel of \( L \) and Range of \( L \).

e) Verify Dimension Theorem (Rank-Nullity Theorem)

7) Let \( \mathbf{x} = (2, -2, 0, -1)^T, \ \mathbf{y} = (1, -1, -1, 1)^T \)

a) Find the angle between \( \mathbf{x} \) and \( \mathbf{y} \).

b) Find the distance between \( \mathbf{x} \) and \( \mathbf{y} \).

c) Verify Cauchy-Schwarz inequality for \( \mathbf{x} \) and \( \mathbf{y} \).

8) Consider the following matrix

\[
A = \begin{pmatrix}
-11 & -5 & -3 \\
12 & 7 & 2 \\
12 & 5 & 4
\end{pmatrix}.
\]

Is \( A \) diagonalizable? If it is find a matrix \( P \) such that \( P^{-1}AP = D \). \((D \) is a diagonal matrix\)

Answer the same question for the following matrix

\[
A = \begin{pmatrix}
5 & 0 & 4 \\
-6 & 1 & -12 \\
-6 & 0 & -5
\end{pmatrix}.
\]

9) a) Find the eigenvalues and eigenvectors of the matrices

\[
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}, \quad \begin{pmatrix}
\cosh \theta & \sinh \theta \\
\sinh \theta & \cosh \theta
\end{pmatrix}
\]

b) Find the \( n \)th powers of the above matrices.

10) Prove that

a) the eigenvalues of a Hermitian matrix are real.

b) the eigenvectors corresponding to distinct eigenvalues of a Hermitian matrix are orthogonal.

11) Prove that the vectors in orthogonal set are linearly independent.

12) Let

\[
A = \begin{pmatrix}
a & b & 0 \\
0 & a & c \\
0 & 0 & a
\end{pmatrix}.
\]

Show that

a) If \( b = 1 \) and \( c = 0 \), then \( A \) has two independent eigenvectors.
b) If $b = c = 1$, then $A$ has one linearly independent eigenvector.
c) If $b = c = 0$, then $A$ has three independent eigenvectors.

13) Show that in an inner product space:
   a) $2||x||^2 + 2||y||^2 = ||x + y||^2 + ||x - y||^2$
   b) $||x + y|| ||x - y|| < ||x||^2 + ||y||^2$
   c) $||x||^2 + ||y||^2 = ||x + y||^2$ if and only if $(x, y) = 0$.
   Interpret these results in terms of the geometry of parallelograms.

14) Let $C[a, b]$ is the space of continuous functions of a variable $t$ in $a \leq t \leq b$. Show that the map $A$ is linear defined by
   a) $Ax = x(t) + \int_a^b K(t, s)x(s)ds$
   b) $Ax = \int_0^t x(s)ds$. 

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