The Solow Growth Model

The **Solow Growth Model** is a model of **capital accumulation** in a pure production economy: there are no prices because we are strictly interested in output = real income. Everyone works all the time, so there is no labor/leisure choice. In fact, there is no choice at all: the consumer always saves a fixed portion of income, always works, and owns the firm so collects all "wage" income and profit in the form of all output. We will not need to model the "consumer".

We assume all people work all the time, and we assume they save, hence invest, a fixed portion of their income. There is no government, hence no taxation nor subsidies; this is a **closed economy**, so there is no trade. Since there are no prices there is no need for money: there are no financial markets, etc.

This model, then, is a model that captures the pure impact savings = investment has on the long run standard of living = per capita income. Since we allow for population growth, this model may be called the **Blue Lagoon Model** (i.e. as opposed to Robinson Crusoe, two people can reproduce).

Ingredients: Consumers and Firms. All consumers own the firms, so consumers receive all output, and therefore all profit and rent.

Aggregates: Output = Real Income = Y_t in period t.
Capital Stock = K_t;
Population Size = N_t = Labor Supply (since everyone works all the time).
Consumption = C_t;
Savings = S_t;
Investment = I_t.

Per Capita: Output = Real Income = $y_t = Y_t/N_t$ in period t.
Capital Stock = $k_t = K_t/N_t$;
Consumption = $c_t = C_t/N_t$;
Savings = $s_t = S_t/N_t$;
Investment = $i_t = I_t/N_t$;

THE POPULATION = THE LABOR SUPPLY

The population grows at a constant rate n:

$$n = \%\Delta N_{t+1} = \frac{N_{t+1} - N_t}{N_t} \Rightarrow N_{t+1} = (1+n)N_t$$

THE FIRM

The firm produces according to a Cobb-Douglas production function:

$$\text{Aggregate}: \; Y_t = AK_t^{\alpha} N_t^{1-\alpha}$$

$$\text{Per Capita}: \; \frac{Y_t}{N_t} = A\frac{K_t^{\alpha} N_t^{1-\alpha}}{N_t}$$

$$y_t = A\frac{K_t^{\alpha} N_t^{1-\alpha}}{N_t^{\alpha} N_t^{1-\alpha}} = A\frac{K_t^{\alpha}}{N_t^{\alpha}}\frac{N_t^{1-\alpha}}{N_t^{1-\alpha}} = A\left(\frac{K_t^{\alpha}}{N_t^{\alpha}}\right) = A\left(\frac{K_t}{N_t}\right)^{\alpha} = Ak_t^{\alpha}$$

$$y_t = Ak_t^{\alpha}$$

THE CONSUMER : CONSUMPTION AND SAVINGS

The consumer saves the fraction s of income:

$$S_t = sY_t \quad \text{and} \quad C_t = (1-s)Y_t \quad \text{hence} \quad Y_t = C_t + S_t$$

Notice this reflects the fact that there is no government (no taxes), and no imports/exports (no trade).

EQUILIBRIUM GROWTH

1. POPULATION

If the population size starts at N_0, then

$$N_t = (1+n)N_{t-1} = (1+n)(1+n)N_{t-2} = \cdots = (1+n)^t N_0$$

We need only know the initial N_0 and the growth rate n to find the population size in any period t.

2. SAVINGS AND INVESTMENT

Since there is no government (no taxes), there are no imports/exports (no trade), consumers receive everything from the firms, and there are no financial markets, savings is simply investment (the only place consumers can put their "money", which is actually output, is simply back into the firm):

$$I_t = S_t$$

3. CAPITAL ACCUMULATION

Aggregate capital grows according to the following **law of motion**:

$$\boxed{K_{t+1} = (1-d)K_t + I_t}$$

Next periods capital stock is this periods discounted for depreciation (d = depreciation rate), plus whatever was invested.

Use the above production function and savings = investment identity to deduce **per capita capital** accumulation evolves according to:

$$\begin{aligned}
K_{t+1} &= (1-d)K_t + I_t \\
&= (1-d)K_t + S_t \\
&= (1-d)K_t + sY_t \\
&= (1-d)K_t + sAK_t^{\alpha}N_t^{1-\alpha}
\end{aligned}$$

$$\frac{K_{t+1}}{N_t} = (1-d)\,\frac{K_t}{N_t} + sA\,\frac{K_t^{\alpha} N_t^{1-\alpha}}{N_t} = (1-d)\,\frac{K_t}{N_t} + sA\,\frac{K_t^{\alpha} N_t^{1-\alpha}}{N_t^{\alpha} N_t^{1-\alpha}}$$

$$\frac{N_{t+1}}{N_t}\,\frac{K_{t+1}}{N_{t+1t}} = (1-d)\,\frac{K_t}{N_t} + sA\,\frac{K_t^{\alpha}}{N_t^{\alpha}}$$

$$(1+n)k_{t+1} = (1-d)k_t + sAk_t^{\alpha}$$

hence

$$\boxed{k_{t+1} = \left(\frac{1-d}{1+n}\right)k_t + \frac{sA}{1+n}\,k_t^{\alpha}}$$

THE STEADY STATE

The per capita capital stock growths, but at a decreasing rate: see below. Eventually growth converges to zero. In this long-run **steady state**

$$k_{t+1} = k_t = k^*$$

We can explicitly solve for k^* as follows:

$$k^* = \left(\frac{1-d}{1+n}\right)k^* + \frac{sA}{1+n}\left(k^*\right)^{\alpha} \Rightarrow 1 - \frac{1-d}{1+n} = sA\left(k^*\right)^{\alpha-1}$$

$$\frac{1+n}{1+n} - \frac{1-d}{1+n} = \frac{sA}{1+n}\left(k^*\right)^{\alpha-1} \Rightarrow \frac{n+d}{1+n} = \frac{sA}{1+n}\,\frac{1}{\left(k^*\right)^{1-\alpha}} \Rightarrow \left(k^*\right)^{1-\alpha} = \frac{sA}{n+d}$$

hence

$$\boxed{\text{Steady State } \; k^* = \left(\frac{sA}{n+d}\right)^{1/(1-\alpha)}}$$

The steady state level of real income, consumption, savings and investment can all be deduced from k^*:

$$y^* = A(k^*)^{\alpha} = A\left(\frac{sA}{n+d}\right)^{\alpha/(1-\alpha)}$$

$$i^* = s^* = sy^* = sA\left(\frac{sA}{n+d}\right)^{\alpha/(1-\alpha)}$$

$$c^* = (1-s)\,y^* = (1-s)\,A\left(\frac{sA}{n+d}\right)^{\alpha/(1-\alpha)}$$

EXAMPLE #1

Consider three economies which differ in their savings rate and/or population growth rate:

1. $n = .02 \quad s = .25 \quad d = .07 \quad \alpha = .75 \quad N_0 = K_0 = A = 1$

2. $n = .08 \quad s = .25 \quad d = .07 \quad \alpha = .75 \quad N_0 = K_0 = A = 1$

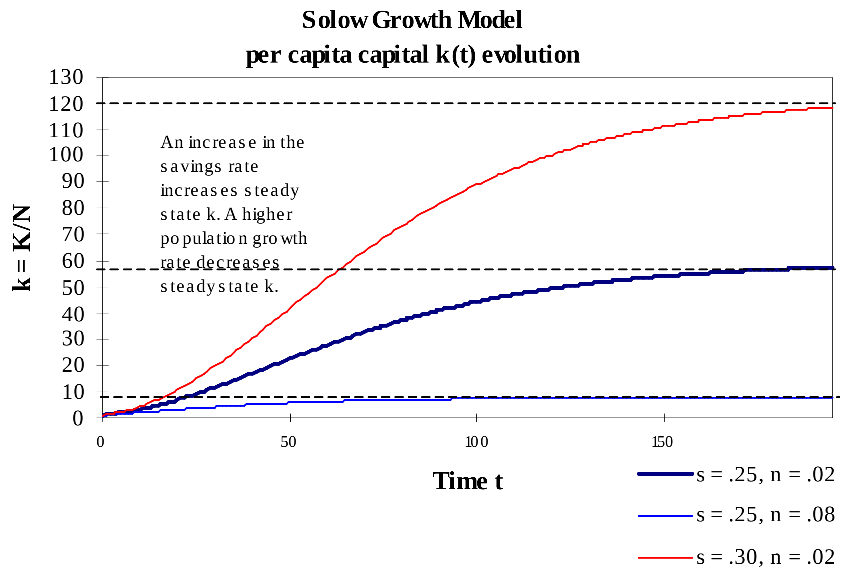
3. $n = .02 \quad s = .30 \quad d = .07 \quad \alpha = .75 \quad N_0 = K_0 = A = 1$

Economy #2 has a high population growth rate; economy #3 has a high savings rate. Per capita capital accumulation and the steady state capital stocks are

1. $k^* = \left(\frac{.25\times 1}{.02+.07}\right)^{1/(1-.75)} = 59.54$

2. $k^* = \left(\frac{.25\times 1}{.08+.07}\right)^{1/(1-.75)} = 7.716$

3. $k^* = \left(\frac{.3\times 1}{.02+.07}\right)^{1/(1-.75)} = 123.46$



1. CAPITAL GROWTH TOWARD THE STEADY STATE

Use the definition of growth

$$\%\Delta k_{t+1} = \frac{k_{t+1} - k_t}{k_t}$$

and the capital accumulation formula

$$k_{t+1} = \left(\frac{1-d}{1+n}\right)k_t + sAk_t^{\alpha}$$

to deduce

$$\begin{aligned}
\%\Delta k_{t+1} &= \frac{k_{t+1} - k_t}{k_t} \\
&= \frac{\left(\frac{1-d}{1+n}\right)k_t - k_t + \frac{sA}{1+n}k_t^{\alpha}}{k_t} = \left(\frac{1-d}{1+n}\right) - 1 + \frac{sA}{1+n}k_t^{\alpha-1} \\
&= \left(\frac{1-d}{1+n}\right) - 1 + \frac{sA}{1+n}k_t^{\alpha-1} = \frac{1-d}{1+n} - \frac{1+n}{1+n} + \frac{sA}{1+n}k_t^{\alpha-1} \\
&= \frac{1-d-1-n}{1+n} + \frac{sA}{1+n}k_t^{\alpha-1} = -\left(\frac{n+d}{1+n}\right) + \frac{sA}{1+n}\frac{1}{k_t^{1-\alpha}}
\end{aligned}$$

Positive Growth

We need to verify that growth is positive as long as the capital stock *k* is less than the long-run steady state level:

$$\text{Growth is positive } \%\Delta k_{t+1} = -\left(\frac{n+d}{1+n}\right) + \frac{sA}{1+n}\frac{1}{k_t^{1-\alpha}} > 0$$

$$\text{when } \frac{sA}{1+n}\frac{1}{k_t^{1-\alpha}} > \frac{n+d}{1+n} \Rightarrow sA\frac{1}{k_t^{1-\alpha}} > n+d \Rightarrow \frac{sA}{n+d} > k_t^{1-\alpha} \Rightarrow \left(\frac{sA}{n+d}\right)^{1/(1-\alpha)} > k_t$$

Growth Declines as *k* Increases

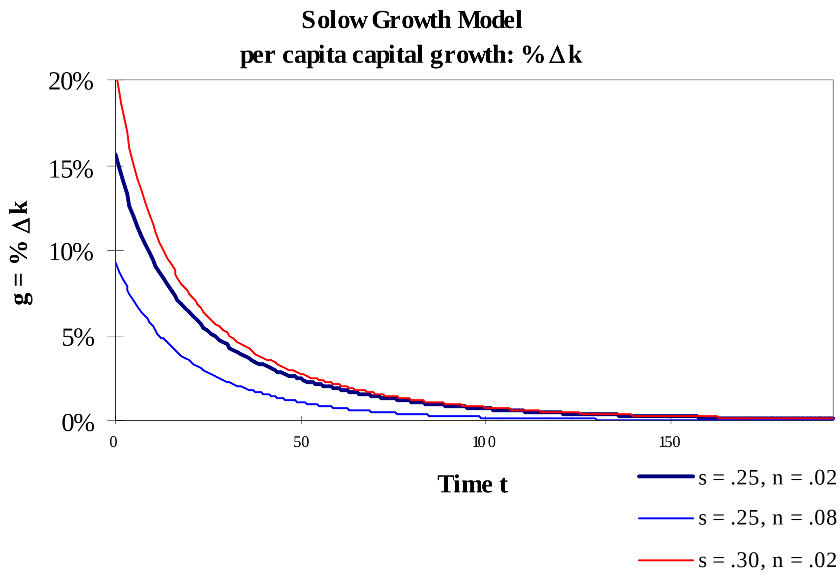
Also, the growth equation shows

$$\%\Delta k_{t+1} = \left\{-\left(\frac{n+d}{1+n}\right) + \frac{sA}{1+n}\frac{1}{k_t^{1-\alpha}\uparrow}\right\}\downarrow$$

In other words, as *k* increases, the growth rate declines. That mathematically verifies the above plots: *k* is clearly increasing, but a decreasing rate.

EXAMPLE #2

The growth plots for Example #1 are:



2. COMPARATIVE STATICS

We can alter the underlying economies in order to inspect what Solow's model predicts about differing countries.

2.1 Higher *n*

The first example clearly shows the for two countries that are identical in every way, except population growth, the higher growth rate leads to slower capital growth and a lower long run level of per capita capital.

2.2 Higher *s*

The first example clearly shows the for two countries that are identical in every way, except the savings rate, the higher savings rate generates a higher capital growth rate and a higher long run level of per capita capital.

2.3 Identical Economies, with Difference Initial k_0

Consider two economies, identical in every way except their initial capital stock per capita:

$$A.\quad n = .03 \quad s = .4 \quad d = .07 \quad \alpha = .5 \quad N_0 = K_0 = A = 1$$

$$B.\quad n = .03 \quad s = .4 \quad d = .07 \quad \alpha = .5 \quad N_0 = A = 1 \quad K_0 = 4$$

Thus, Economy B starts with four times the level of capital per capita as Economy A.

Notice

$$k_1^A = \left(\frac{1-d}{1+n}\right)k_0^A + \frac{sA}{1+n}\left(k_0^A\right)^\alpha$$

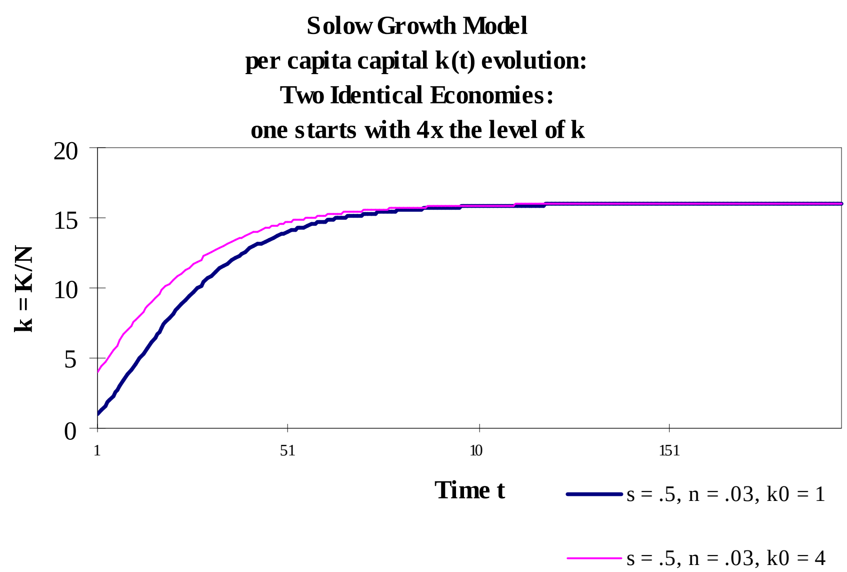
$$k_1^B = \left(\frac{1-d}{1+n}\right)k_0^B + \frac{sA}{1+n}\left(k_0^B\right)^\alpha = 4\left(\frac{1-d}{1+n}\right)k_0^A + \frac{sA}{1+n}4^\alpha\left(k_0^A\right)^\alpha > \left(\frac{1-d}{1+n}\right)k_0^A + \frac{sA}{1+n}\left(k_0^A\right)^\alpha = k_1^A$$

because $4^\alpha > 1$ for any $0 < \alpha < 1$. And so on.

Economy B will (almost) always have more capital. Why "*almost*"? Because their steady-states are identical. The *steady state does not depend on initial conditions*, and only on technology, saving and population growth parameters.

Both have

$$k^* = \left(\frac{sA}{n+d}\right)^{1/(1-\alpha)} = \left(\frac{.4\times 1}{.03+.07}\right)^{1/(1-.5)} = \left(\frac{.4}{.1}\right)^2 = 16$$



Thus, even economies that are otherwise identical, one can never **take-over** the other.

3. THE GOLDEN RULE OF SAVINGS

Although a higher savings rate generates more per capita capital with which to generate more output, a constant increase in *s* cannot always lead to a higher level of per capita consumption. Clearly a very high savings rate implies people are consuming very little of that very large amount of output that is being produced.

Imagine very hard working people, but very hungry, planting a corn crop, and harvesting the corn. Instead of eating a lot they plant 80% of the kernels. The result is an enormous crop the next season. They again eat very little, invest 80% of the output toward production, and so on. The total level of output per capital sky-rockets, but is welfare truly higher? If we measure welfare by consumption clearly it is not.

The steady state level of consumption is

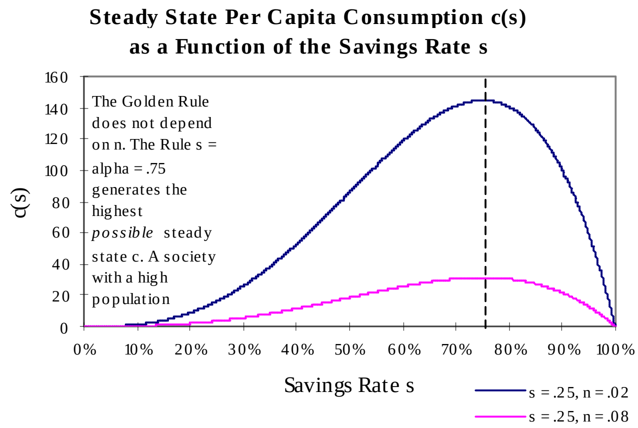
$$\text{Steady State } c^* = (1-s)y^* = (1-s)\left(k^*\right)^{\alpha}$$
$$= (1-s)\left(\frac{sA}{n+d}\right)^{\alpha/(1-\alpha)}$$
$$c^* = (1-s)s^{\alpha/(1-\alpha)}\left(\frac{A}{n+d}\right)^{\alpha/(1-\alpha)}$$

Thus, the long-run level of per capita consumption is a function of the savings rate s:

$$\boxed{c^*(s) = (1-s)s^{\alpha/(1-\alpha)}\left(\frac{A}{n+d}\right)^{\alpha/(1-\alpha)}}$$

EXAMPLE #3

Consider Economies #1 and #2 in Example #1. We can plot $^*c(s)$ as a function of s:



There is a clear tug-of-war: a higher s means less consumption, but allows for higher long-run capital accumulation and therefore higher steady state income. The optimal savings rate is the **Golden Rule**: the *savings rate the optimizes steady state per capita consumption*:

$$\max_{0\le s\le 1} c^*(s) = \max_{0\le s\le 1}(1-s)s^{\alpha/(1-\alpha)}\left(\frac{A}{n+d}\right)^{\alpha/(1-\alpha)}$$

$$\text{FOC}_s : \frac{\partial}{\partial} c^*(s) = 0$$
$$: -s^{\alpha/(1-\alpha)}\left(\frac{A}{n+d}\right)^{\alpha/(1-\alpha)} + \frac{\alpha}{1-\alpha}(1-s)s^{\alpha/(1-\alpha)-1}\left(\frac{A}{n+d}\right)^{\alpha/(1-\alpha)} = 0$$
$$: -1 + \frac{\alpha}{1-\alpha}(1-s)s^{-1} = 0$$

$$:\frac{\alpha}{1-\alpha}(1-s)=s$$
$$:\alpha(1-s)=s(1-\alpha)$$
$$:\alpha=s(1-\alpha)+\alpha s=s$$

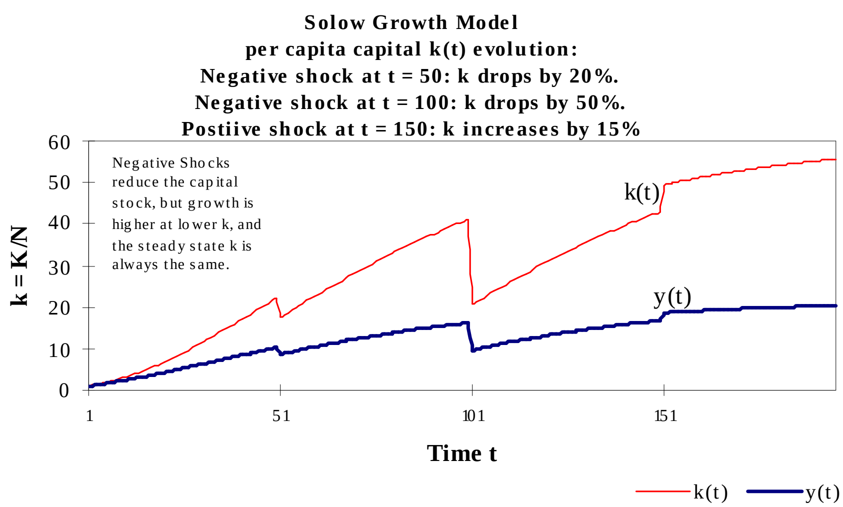
$$\text{GOLDEN RULE}: s=\alpha$$

Compare the result to the plots above. For two otherwise identical economies with the same α = .75, the Golden Rule is s = .75, and indeed the steady state per capita consumption is optimizes in both cases identically at s = .75.

EXAMPLE #4

Although the Solow Model does not itself build in any sense of a "business cycle", or suggest anything like a shock (all plots clearly show the model predicts perfectly uniform growth!), we ourselves can introduce a shock at any time.

Consider a scenario where an economy experiences two negative shocks, one small and one large, and a medium positive shock. The small negative shock (at t = 50) is causes by news that a sector (e.g. the energy sector) has been substantially over invested, profits are predicted to be negative, and therefore investment capital is withdrawn: k falls by 20%. The large negative shock (at t = 100) is due to a massive storm: k falls by 50%. The positive shock (at t = 150) is due to a capital infusion by the International Monetary Fund: k instantly increases by 15%.



The Solow Model is a very simple model in the final analysis: at whatever point or state the economy is in, growth immediately occurs (fast if k is small, slow if k is large), while the standard of living y slowly approaches the long-run steady state.

SOLOW MODEL AND REALITY

The Solow Model predicts growth is always positive, but slowly declines to zero. Economies with a high population growth rate can never take-over. In fact, otherwise identical economies where one simply starts with a smaller level of per capita capital, will never take-over to the other economy.

Nevertheless, the Solow Model does correctly predict that higher population growth rates, and lower savings = investment rates are associated with lower growth levels, and lower standards of living.