DYNAMICS OF RECIPROCATING ENGINES
Indicator Diagrams

1. **4-Stroke Otto engine**
   - Pressure
   - High increase in pressure
   - Ignition
   - Expansion
   - Compression
   - Exhaust
   - Suction (Gasoline+Air)
   - STROKE

2. **4-Stroke Diesel engine**
   - Pressure
   - Fuel injection
   - Expansion
   - Compression
   - Exhaust
   - Suction (Air)
   - STROKE

3. **2-Stroke Diesel engine**
   - Pressure
   -Expansion
   - Compression
   - Suction
   - Exhaust
   - STROKE
In-Line Engines

FIGURE 18-1
A three-cylinder in-line engine: (a) front view; (b) side view; (c) firing order.

FIGURE 18-2
V Engines

FIGURE 18-3
Crank arrangements of V engines: (a) single crank per pair of cylinders—connecting rods interlock with each other and are of fork-and-blade design; (b) single crank per pair of cylinders—the master connecting rod carries a bearing for the articulated rod; (c) separate crank throws connect to staggered rods and pistons.
GAS FORCES

Binomial Theorem:

\[ \sqrt{1 - x} = 1 - \frac{x}{2} + \frac{x^2}{8} - \frac{x^3}{16} + \frac{5x^4}{128} + \ldots \]

\[ \sqrt{1 - \left(\frac{r}{\ell}\sin\omega t\right)^2} = 1 - \frac{1}{2}\left(\frac{r}{\ell}\sin\omega t\right)^2 \frac{1}{8} + \frac{1}{8}\left(\frac{r}{\ell}\sin\omega t\right)^4 - \ldots \]

So,

Since in most engines, \( \frac{r}{\ell} \) is so small, and only the first two terms of the series may be taken with sufficient accuracy.

Therefore,

We can write

\[ x = r \cos\omega t + \ell \left[ 1 - \frac{r^2}{2\ell^2} \left( \frac{1 - \cos 2\omega t}{2} \right) \right] \]

and finally

\[ x = \ell - \frac{r^2}{4\ell} + r(\cos\omega t + \frac{r}{4\ell}\cos 2\omega t) \]
It is important to note that always angular velocity ($\omega$) is assumed constant. Therefore, acceleration of the piston may be obtained as,

$$\ddot{x} = -r\omega^2 \left( \cos\omega t + \frac{r}{\ell} \cos2\omega t \right)$$

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$$F_{14} = P \tan \phi j$$

$$\tan \phi = \frac{\sin \phi}{\cos \phi} = \frac{(r/\ell) \sin \omega t}{\sqrt{1 - \left(\frac{r}{\ell} \sin \omega t\right)^2}}$$

$$\tan \phi = \frac{r}{\ell} \sin \omega t \left( 1 + \frac{r^2}{2\ell^2} \sin^2 \omega t \right)$$

Using Binomial theorem for the denominator of this ratio and taking only first two terms of the series,

Therefore,

$$F_{14} = P \left( \frac{r}{\ell} \sin \omega t \left( 1 + \frac{r^2}{2\ell^2} \sin^2 \omega t \right) \right)$$

Similarly,

$$F_{34} = \frac{P}{\cos \phi} = \frac{P}{\sqrt{1 - \left(\frac{r}{\ell} \sin \omega t\right)^2}} = P \left( 1 + \frac{r^2}{2\ell^2} \sin^2 \omega t \right)$$

$$F_{34} = Pi - F_{14}j = Pi - P \tan \phi j$$
\[ T_{12}k + xi \times (F_{14}j) = 0 \]

\[ T_{12} = -F_{14} x k = -P \tan \phi \times k \]

\[ T_{21} = P \tan \phi \times k \]

Making appropriate substitutions and neglecting higher order terms, \textit{torque delivered by the crank to the shaft} is obtained

\[ T_{21} = Pr \sin \omega t \left(1 + \frac{r}{l} \cos \omega t\right) \hat{k} \]
EQUIVALENT MASSES

1) Total masses should be the same:
2) The mass center positions should be the same:
(If \( r_{G3} = 0 \), \( m_{3A} \ell_A = m_{3B} \ell_B \))

\[
m_{3A} = m_3 \frac{\ell_B}{\ell} \quad m_{3B} = m_3 \frac{\ell_A}{\ell}
\]

\[
m_3 = m_{3A} + m_{3B}
\]

\[
r_{G3} = \frac{-\ell_A m_{3A} + \ell_B m_{3B}}{m_3}
\]

In practice, for a connecting rod

\[
m_{3A} \approx \frac{2}{3} m_3 \quad m_{3B} \approx \frac{1}{3} m_3
\]
EQUIVALENT MASSES

• The first two conditions is called STATIC EQUIVALENCE.

• 3) The mass moment of inertias should be the same: \[ I_{G_3} = m_{3A} \ell_A^2 + m_{3B} \ell_B^2 = m_3 \ell_A \ell_B \]

• The third condition is called DYNAMIC EQUIVALENCE.

• For a connecting rod of a slider-crank mechanism, our aim is to divide the total mass into two masses; one, at point A (rotating) and the other one, at point B (reciprocating). The third condition does not conform our aim, therefore it is not satisfied throughout the analysis of connecting rod.
Dividing the crank mass into two parts, at $O_2$ and A, regarding the static equivalence conditions.

\[ m_2 r_G = m_2 A r + 0 \quad \text{or} \quad m_2 A = m_2 \frac{r_G}{r} \]
INERTIA FORCES

\[ m_A = m_{2A} + m_{3A} \]
\[ m_B = m_{3B} + m_4 \]

\[ R_A = r \cos \omega t \hat{i} + r \sin \omega t \hat{j} \]

Constant angular velocity assumption for crank gives the acceleration of point A as,

\[ a_A = -r \omega^2 \cos \omega t \hat{i} - r \omega^2 \sin \omega t \hat{j} \]
\[ -m_A a_A = m_A r \omega^2 \cos \omega t \hat{i} + m_A r \omega^2 \sin \omega t \hat{j} \]

Acceleration of the piston has been found as,

\[ a_B = -r \omega^2 \left( \cos \omega t + \frac{r}{l} \cos 2 \omega t \right) \hat{i} \]
\[ -m_B a_B = m_B r \omega^2 \left( \cos \omega t + \frac{r}{l} \cos 2 \omega t \right) \hat{i} \]

The components of total inertia force,

\[ F_x = (m_A + m_B) r \omega^2 \cos \omega t + \left( m_B - \frac{r}{l} \right) r \omega^2 \cos 2 \omega t \]
\[ F_y = m_A r \omega^2 \sin \omega t \]
Inertia Torque

\[ T_{12}k + xi X (-F_{14}j) = 0 \]

\[ T_{12} = F_{14} x k = -m_B a_B \tan \phi x k \]

\[ T_{21} = -(-m_B \ddot{x} \tan \phi) x \hat{k} \]

Making appropriate substitutions, neglecting higher order terms and using trigonometric identities, inertia torque exerted by the engine on the shaft is obtained,

\[ T_{21} = \frac{m_B \ddot{x}}{2} r^2 \omega^2 \left( \frac{r}{2l} \sin \omega t - \sin 2\omega t - \frac{3r}{2l} \sin 3\omega t \right) k \]

Inertia torque is a periodic function, including the first three harmonics.
BEARING LOADS IN THE SINGLE-CYLINDER ENGINE

• The resultant bearing loads in a single-cylinder engine are made up of due to the following four forces:

• 1) Gas force (examined at the beginning).
• 2) Inertia force due to the mass of piston $m_4$.
• 3) Inertia force due to the mass of connecting rod $m_{3b}$ at piston-pin end.
• 4) Inertia force due to the mass of connecting rod $m_{3a}$ at crank-pin end.

• (It is assumed that crank has been balanced.)
2) Inertia force due to the mass of piston $m_4$
3) Inertia force due to the mass of connecting rod \( m_{3B} \) at piston-pin end
4) Inertia force due to the mass of connecting rod $m_{3A}$ at crank-pin end

\[ F_{34} = F_{14} = 0 \]

\[ F'''' = -F_A^a = -(m_{3A}a_A) = m_{3A}a_A = F''''_{12} \]

\[ F''''_{23} = m_{3A}r\omega^2 = F''''_{12} \]

If a counterweight of $m_{3A}$ is added at point C, no bearing force will act at $O_2$. Therefore shaking force due to $m_{3A}$ is eliminated.
Superposition

\[ F_{41} = F'_{41} + F''_{41} + F'''_{41} \]
\[ F_{34} = F'_{34} + F''_{34} + F'''_{34} \]
\[ F_{32} = F'_{32} + F''_{32} + F'''_{32} + F''''_{32} \]
\[ F_{32} = [m_{3A} r \omega^2 \cos \omega t - (m_{3B} + m_4) \ddot{x} - P] i + [m_{3A} r \omega^2 \sin \omega t + [(m_{3B} + m_4) \ddot{x} + P] \tan \theta] j \]
\[ F_{21} = F_{32} \]
\[ F_{41} = -[(m_{3B} + m_4) \ddot{x} + P] \tan \theta j \]
\[ F_{34} = (m_4 \ddot{x} + P) i - [(m_{3B} + m_4) \ddot{x} + P] \tan \theta j \]
CRANKSHAFT TORQUE

\[ \sum M_{O2} = 0 \]

\[ T_{21} + x_i \times F_{41}j = 0 \]

\[ T_{21} = -F_{41}xk = \left[ (m_{3B} + m_4)\ddot{x} + P \right]xtan\phi k \]

The torque delivered by the crankshaft to the load.
ENGINE SHAKING FORCES
(due to only reciprocating masses)

Shaking Force: \[ F_{21}^x = -m_B a_B \] (Linear vibration in x direction)

Shaking Couple: \[ T_{21} = x F_{41} \] (Torsional vibration about crank center)

Graphically, \[ F_{21}^x = OA' + OB' \]

\[ F_{21}^x = -m_B A_B = -m_B \left[ -r \omega^2 (\cos \omega t + \frac{r}{\ell} \cos 2\omega t) \right] i \]

\[ F_{21}^x = m_B r \omega^2 \cos \omega t + m_B r \omega^2 \frac{r}{\ell} \cos 2\omega t \]
PROBLEM

Stroke = 2r = 90 mm → r = 45 mm

d = 100 mm → A = \pi d^2 / 4 = \pi 100^2 / 4 = 7853.98 \text{ mm}^2 = 7.854 \times 10^{-3} \text{ m}^2

n = 4400 \text{ rpm} → \omega = 4400 (2\pi) / 60 = 460.767 \text{ r/s}

m_{3A} = 0.8 \text{ kg} \quad m_{3B} = 0.38 \text{ kg} \quad m_4 = 1.64 \text{ kg}

Obtain cylinder pressure for the 30% expansion case.

\[ p = 1.857 \text{ Mpa} = 1857000 \text{ Pa} (\text{N/m}^2) \]

\[ P = 1857000 (7.854 \times 10^{-3}) = 14584.878 \text{ N} \]
Force Calculations

\[ F_{41} = -[(m_B + m_4)\ddot{x} + P] \tan \theta j \]

\[ F_{41} = -(0.38 + 1.64)(-3578.663) + 14584.878)(0.1155)j \]

\[ F_{41} = -849.616j \ N \]

\[ F_{34} = (m_4 \ddot{x} + P)i - [(m_B + m_4)\ddot{x} + P] \tan \theta j \]

\[ F_{34} = [1.64(-3578.663) + 14584.878]i - 849.616j = 8715.87i - 849.616j \ N \]

\[ F_{32} = [m_A r \omega^2 \cos \omega t - (m_B + m_4)\ddot{x} - P]i + [m_A r \omega^2 \sin \omega t + [(m_B + m_4)\ddot{x} + P] \tan \theta]j \]

\[ F_{32} = [0.8 \left(\frac{45}{1000}\right) 460.767^2 \cos 63.2 - (0.38 + 1.64)(-3578.663) - 14584.878]i \]
\[ + \left[0.8 \left(\frac{45}{1000}\right) 460.767^2 \sin 63.2 \right. \]
\[ + [(0.38 + 1.64)(-3578.663) + 14584.878] \times 0.1155 \]j

\[ F_{32} = -3909.91i + 7671.67j \ N \]

\[ F_{21} = F_{32} \]

\[ T_{21} = -F_{41}xk = [(m_B + m_4)\ddot{x} + P]xtan \theta k \]

\[ T_{21} = -(-849.616)(0.368)k \]

\[ T_{21} = 312.66 \ k \ N.m \]