Overview

- Part 1 – Gate Circuits and Boolean Equations
  - Binary Logic and Gates
  - Boolean Algebra
  - Standard Forms
- Part 2 – Circuit Optimization
  - Two-Level Optimization
  - Map Manipulation
  - Practical Optimization (Espresso)
  - Multi-Level Circuit Optimization
- Part 3 – Additional Gates and Circuits
  - Other Gate Types
  - Exclusive-OR Operator and Gates
  - High-Impedance Outputs
Binary Logic and Gates

- **Binary variables** take on one of two values.
- **Logical operators** operate on binary values and binary variables.
- Basic logical operators are the **logic functions** AND, OR and NOT.
- **Logic gates** implement logic functions.
- **Boolean Algebra**: a useful mathematical system for specifying and transforming logic functions.
- We study Boolean algebra as a foundation for designing and analyzing digital systems!

Binary Variables

- Recall that the two binary values have different names:
  - True/False
  - On/Off
  - Yes/No
  - 1/0
- We use 1 and 0 to denote the two values.
- Variable identifier examples:
  - A, B, y, z, or X₁ for now
  - RESET, START_IT, or ADD1 later
Logical Operations

- The three basic logical operations are:
  - AND
  - OR
  - NOT

- AND is denoted by a dot (·).
- OR is denoted by a plus (+).
- NOT is denoted by an overbar (¯), a single quote mark (') after, or (~) before the variable.

Notation Examples

- Examples:
  - \( Y = A \cdot B \) is read “\( Y \) is equal to \( A \) AND \( B \).”
  - \( z = x + y \) is read “\( z \) is equal to \( x \) OR \( y \).”
  - \( X = \overline{A} \) is read “\( X \) is equal to NOT \( A \).”

- Note: The statement:
  \( 1 + 1 = 2 \) (read “one plus one equals two”)
  is not the same as
  \( 1 + 1 = 1 \) (read “1 or 1 equals 1”).
Operator Definitions

- Operations are defined on the values "0" and "1" for each operator:

<table>
<thead>
<tr>
<th>AND</th>
<th>OR</th>
<th>NOT</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 \times 0 = 0</td>
<td>0 + 0 = 0</td>
<td>\overline{0} = 1</td>
</tr>
<tr>
<td>0 \times 1 = 0</td>
<td>0 + 1 = 1</td>
<td>\overline{1} = 0</td>
</tr>
<tr>
<td>1 \times 0 = 0</td>
<td>1 + 0 = 1</td>
<td></td>
</tr>
<tr>
<td>1 \times 1 = 1</td>
<td>1 + 1 = 1</td>
<td></td>
</tr>
</tbody>
</table>

Truth Tables

- *Truth table* – a tabular listing of the values of a function for all possible combinations of values on its arguments

- Example: Truth tables for the basic logic operations:

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
<th>Z = X \cdot Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
<th>Z = X + Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>X</th>
<th>Z = \overline{X}</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>
Logic Function Implementation

- **Using Switches**
  - For inputs:
    - logic 1 is switch closed
    - logic 0 is switch open
  - For outputs:
    - logic 1 is light on
    - logic 0 is light off.
  - NOT uses a switch such that:
    - logic 1 is switch open
    - logic 0 is switch closed

Logic Function Implementation (Continued)

- **Example: Logic Using Switches**

- Light is on (L = 1) for
  \[ L(A, B, C, D) = \]
  and off (L = 0), otherwise.

- Useful model for relay circuits and for CMOS gate circuits, the foundation of current digital logic technology
Logic Gates

- In the earliest computers, switches were opened and closed by magnetic fields produced by energizing coils in relays. The switches in turn opened and closed the current paths.
- Later, vacuum tubes that open and close current paths electronically replaced relays.
- Today, transistors are used as electronic switches that open and close current paths.
- Optional: Chapter 6 – Part 1: The Design Space

Logic Gate Symbols and Behavior

- Logic gates have special symbols:

(a) Graphic symbols

(b) Timing diagram

And waveform behavior in time as follows:

<table>
<thead>
<tr>
<th>X</th>
<th>0</th>
<th>0</th>
<th>1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Y</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(AND) X \cdot Y</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(OR) X</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(NOT) X</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Gate Delay

- In actual physical gates, if one or more input changes causes the output to change, the output change does not occur instantaneously.
- The delay between an input change(s) and the resulting output change is the gate delay denoted by $t_G$:

```
+---+---+---+---+
|   |   |   |   |
| I | O | I | O |
| 0 | 1 | 1 | 0 |
| 1 | 0 | 0 | 1 |
```

$t_G = 0.3 \text{ ns}$

Logic Diagrams and Expressions

- Boolean equations, truth tables and logic diagrams describe the same function!
- Truth tables are unique; expressions and logic diagrams are not. This gives flexibility in implementing functions.
Boolean Algebra

- An algebraic structure defined on a set of at least two elements, B, together with three binary operators (denoted +, ·, and ¬) that satisfies the following basic identities:

1. \( X + 0 = X \)
2. \( X \cdot 1 = X \)
3. \( X + 1 = 1 \)
4. \( X \cdot 0 = 0 \)
5. \( X + X = X \)
6. \( X \cdot X = X \)
7. \( X + \overline{X} = 1 \)
8. \( X \cdot \overline{X} = 0 \)
9. \( \overline{X} = X \)
10. \( X + Y = Y + X \)
11. \( XY = YX \)
12. \( (X + Y) + Z = X + (Y + Z) \)
13. \( (XY)Z = X(YZ) \)
14. \( X(Y + Z) = XY + XZ \)
15. \( X + YZ = (X + Y)(X + Z) \)
16. \( \overline{X + Y} = \overline{X} \cdot \overline{Y} \)
17. \( \overline{X \cdot Y} = \overline{X + Y} \)

The identities above are organized into pairs. These pairs have names as follows:

- 1-4 Existence of 0 and 1
- 5-6 Idempotence
- 7-8 Existence of complement
- 9 Involution
- 10-11 Commutative Laws
- 12-13 Associative Laws
- 14-15 Distributive Laws
- 16-17 DeMorgan’s Laws

Some Properties of Identities & the Algebra

- If the meaning is unambiguous, we leave out the symbol “·”
- The identities above are organized into pairs. These pairs have names as follows:

  1-4 Existence of 0 and 1
  5-6 Idempotence
  7-8 Existence of complement
  9 Involution
  10-11 Commutative Laws
  12-13 Associative Laws
  14-15 Distributive Laws
  16-17 DeMorgan’s Laws

- The dual of an algebraic expression is obtained by interchanging + and · and interchanging 0’s and 1’s.
- The identities appear in dual pairs. When there is only one identity on a line the identity is self-dual, i.e., the dual expression = the original expression.
Some Properties of Identities & the Algebra (Continued)

- Unless it happens to be self-dual, the dual of an expression does not equal the expression itself.
  
  Example: \( F = (A + \overline{C}) \cdot B + 0 \)
  
  dual \( F = (A \cdot \overline{C} + B) \cdot 1 = A \cdot \overline{C} + B \)

- Example: \( G = X \cdot Y + (W + Z) \)
  
  dual \( G = \)

- Example: \( H = A \cdot B + A \cdot C + B \cdot C \)
  
  dual \( H = \)

- Are any of these functions self-dual?

Some Properties of Identities & the Algebra (Continued)

- There can be more that 2 elements in \( B \), i.e., elements other than 1 and 0. What are some common useful Boolean algebras with more than 2 elements?
  
  1. Algebra of Sets
  2. Algebra of \( n \)-bit binary vectors

- If \( B \) contains only 1 and 0, then \( B \) is called the switching algebra which is the algebra we use most often.
Boolean Operator Precedence

- The order of evaluation in a Boolean expression is:
  1. Parentheses
  2. NOT
  3. AND
  4. OR
- Consequence: Parentheses appear around OR expressions
- Example: $F = A(B + C)(C + \overline{D})$

Example 1: Boolean Algebraic Proof

- $A + A\cdot B = A$ (Absorption Theorem)

Proof Steps | Justification (identity or theorem)
--- | ---
$A + A\cdot B$ | $X = X \cdot 1$
$= A \cdot 1 + A \cdot B$ | $X \cdot (1 + Y) = X \cdot (1 + Z)$
$= A \cdot (1 + B)$ | Distributive Law
$= A \cdot 1$ | $1 + X = 1$
$= A$ | $X \cdot 1 = X$

- Our primary reason for doing proofs is to learn:
  - Careful and efficient use of the identities and theorems of Boolean algebra, and
  - How to choose the appropriate identity or theorem to apply to make forward progress, irrespective of the application.
Example 2: Boolean Algebraic Proofs

- \( AB + \overline{AC} + BC = AB + \overline{AC} \) (Consensus Theorem)

Proof Steps Justification (identity or theorem)

\[
\begin{align*}
AB + \overline{AC} + BC & = AB + \overline{AC} + 1 \cdot BC \\
& = AB + \overline{AC} + (A + A) \cdot BC \\
& = AB + \overline{AC} + (A + \overline{A}) \cdot BC \\
& = \\
\end{align*}
\]

Example 3: Boolean Algebraic Proofs

- \( (X + Y)Z + XY = \overline{Y}(X + Z) \)

Proof Steps Justification (identity or theorem)

\[
\begin{align*}
(X + \overline{Y})Z + X \overline{Y} & = \\
\end{align*}
\]
Useful Theorems

- $x \cdot y + \bar{x} \cdot y = y$ \quad (x + y)(\bar{x} + y) = y \quad \text{Minimization}
- $x + x \cdot y = x$ \quad $x \cdot (x + y) = x \quad \text{Absorption}$
- $x + \bar{x} \cdot y = x + y$ \quad $x \cdot (\bar{x} + y) = x \cdot y \quad \text{Simplification}$
- $x \cdot y + \bar{x} \cdot z + y \cdot z = x \cdot y + \bar{x} \cdot z$ \quad $\text{Consensus}$
- $(x + y) \cdot (\bar{x} + z) \cdot (y + z) = (x + y) \cdot (\bar{x} + z)$
- $\bar{x + y} = \bar{x} \cdot \bar{y}$ \quad $\overline{x \cdot y} = \bar{x} + \bar{y} \quad \text{DeMorgan's Laws}$

Proof of Simplification

$x \cdot y + \bar{x} \cdot y = y$ \quad $(x + y)(\bar{x} + y) = y$
Proof of DeMorgan’s Laws

\[ x + y = \overline{x} \cdot \overline{y} \quad x \cdot y = \overline{x} + \overline{y} \]

Boolean Function Evaluation

\[
\begin{array}{c|c|c|c|c|c|c}
\text{F1} = x\overline{y}z \\
\text{F2} = x + \overline{y}z \\
\text{F3} = \overline{x}\overline{y}z + \overline{x} y z + x\overline{y} \\
\text{F4} = x\overline{y} + \overline{x} z \\
\hline
x & y & z & F1 & F2 & F3 & F4 \\
\hline
0 & 0 & 0 & 0 & 0 & - & - \\
0 & 0 & 1 & 0 & 0 & 1 & - \\
0 & 1 & 0 & 0 & 0 & - & 0 \\
0 & 1 & 1 & 0 & 0 & - & - \\
1 & 0 & 0 & 0 & 0 & 1 & - \\
1 & 0 & 1 & 0 & 0 & 1 & - \\
1 & 1 & 0 & 1 & 1 & - & - \\
1 & 1 & 1 & 0 & 1 & - & - \\
\end{array}
\]
Expression Simplification

- An application of Boolean algebra
- Simplify to contain the smallest number of literals (complemented and uncomplemented variables):
  \[ AB + ACD + \overline{A}BD + \overline{A}C\overline{D} + ABD + CD \]
  \[ = AB + ABCD + AC + \overline{A}CD + \overline{A}BD \]
  \[ = AB + AB(CD) + AC(D + \overline{D}) + B\overline{D} \]
  \[ = AB + A + BD + B\overline{D} + \overline{A}C \]
  \[ = B(A + D) + \overline{A}C \quad 5 \text{ literals} \]

Complementing Functions

- Use DeMorgan's Theorem to complement a function:
  1. Interchange AND and OR operators
  2. Complement each constant value and literal
- Example: Complement \( F = \overline{x}y\overline{z} + xy\overline{z} \)
  \( \overline{F} = (x + \overline{y} + z)(\overline{x} + y + z) \)
- Example: Complement \( G = (\overline{a} + bc)d + e \)
  \( \overline{G} = \)
Overview – Canonical Forms

- What are Canonical Forms?
- Minterms and Maxterms
- Index Representation of Minterms and Maxterms
- Sum-of-Minterm (SOM) Representations
- Product-of-Maxterm (POM) Representations
- Representation of Complements of Functions
- Conversions between Representations

Canonical Forms

- It is useful to specify Boolean functions in a form that:
  - Allows comparison for equality.
  - Has a correspondence to the truth tables
- Canonical Forms in common usage:
  - Sum of Minterms (SOM)
  - Product of Maxterms (POM)
Minterms

- **Minterms** are AND terms with every variable present in either true or complemented form.
- Given that each binary variable may appear normal (e.g., \(x\)) or complemented (e.g., \(\overline{x}\)), there are \(2^n\) minterms for \(n\) variables.
- **Example:** Two variables (X and Y) produce
  
  \[2 \times 2 = 4\] combinations:
  
  - \(XY\) (both normal)
  - \(X\overline{Y}\) (X normal, Y complemented)
  - \(X\overline{Y}\) (X complemented, Y normal)
  - \(\overline{X}\overline{Y}\) (both complemented)
- Thus there are **four minterms** of two variables.

Maxterms

- **Maxterms** are OR terms with every variable in true or complemented form.
- Given that each binary variable may appear normal (e.g., \(x\)) or complemented (e.g., \(\overline{x}\)), there are \(2^n\) maxterms for \(n\) variables.
- **Example:** Two variables (X and Y) produce
  
  \[2 \times 2 = 4\] combinations:
  
  - \(X + Y\) (both normal)
  - \(X + \overline{Y}\) (x normal, y complemented)
  - \(\overline{X} + Y\) (x complemented, y normal)
  - \(\overline{X} + \overline{Y}\) (both complemented)
Maxterms and Minterms

- Examples: Two variable minterms and maxterms.

<table>
<thead>
<tr>
<th>Index</th>
<th>Minterm</th>
<th>Maxterm</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\overline{x} \overline{y}$</td>
<td>$x + y$</td>
</tr>
<tr>
<td>1</td>
<td>$\overline{x} y$</td>
<td>$x + \overline{y}$</td>
</tr>
<tr>
<td>2</td>
<td>$x \overline{y}$</td>
<td>$\overline{x} + y$</td>
</tr>
<tr>
<td>3</td>
<td>$x y$</td>
<td>$\overline{x} + \overline{y}$</td>
</tr>
</tbody>
</table>

- The index above is important for describing which variables in the terms are true and which are complemented.

Standard Order

- Minterms and maxterms are designated with a subscript
- The subscript is a number, corresponding to a binary pattern
- The bits in the pattern represent the complemented or normal state of each variable listed in a standard order.
- All variables will be present in a minterm or maxterm and will be listed in the same order (usually alphabetically)
- Example: For variables a, b, c:
  - Maxterms: $(a + b + \overline{c}), (a + b + c)$
  - Terms: $(b + a + c), a \overline{c} b$, and $(c + b + a)$ are NOT in standard order.
  - Minterms: $a \overline{b} c, a \overline{b} c, \overline{a} \overline{b} c$
  - Terms: $(a + c), \overline{b} c$, and $(\overline{a} + b)$ do not contain all variables
Purpose of the Index

- The index for the minterm or maxterm, expressed as a binary number, is used to determine whether the variable is shown in the true form or complemented form.

- For Minterms:
  - “1” means the variable is “Not Complemented” and
  - “0” means the variable is “Complemented”.

- For Maxterms:
  - “0” means the variable is “Not Complemented” and
  - “1” means the variable is “Complemented”.

Index Example in Three Variables

- Example: (for three variables)
- Assume the variables are called X, Y, and Z.
- The standard order is X, then Y, then Z.
- The Index 0 (base 10) = 000 (base 2) for three variables). All three variables are complemented for minterm 0 ( \( \overline{X}, \overline{Y}, \overline{Z} \)) and no variables are complemented for Maxterm 0 (X,Y,Z).
  - Minterm 0, called \( m_0 \) is \( XYZ \).
  - Maxterm 0, called \( M_0 \) is \( X + Y + Z \).
  - Minterm 6?
  - Maxterm 6?
Index Examples – Four Variables

<table>
<thead>
<tr>
<th>Index</th>
<th>Binary</th>
<th>Minterm</th>
<th>Maxterm</th>
</tr>
</thead>
<tbody>
<tr>
<td>i</td>
<td>Pattern</td>
<td>mi</td>
<td>Mi</td>
</tr>
<tr>
<td>0</td>
<td>0000</td>
<td>$\bar{a}\bar{b}\bar{c}\bar{d}$</td>
<td>$a+b+c+d$</td>
</tr>
<tr>
<td>1</td>
<td>0001</td>
<td>$a\ b\ c\ d$</td>
<td>?</td>
</tr>
<tr>
<td>3</td>
<td>0011</td>
<td>?</td>
<td>$a+b+\bar{c}+\bar{d}$</td>
</tr>
<tr>
<td>5</td>
<td>0101</td>
<td>$\bar{a}\ b\ \bar{c}\ d$</td>
<td>$a+\bar{b}+c+\bar{d}$</td>
</tr>
<tr>
<td>7</td>
<td>0111</td>
<td>?</td>
<td>$a+b+\bar{c}+\bar{d}$</td>
</tr>
<tr>
<td>10</td>
<td>1010</td>
<td>$\bar{a}\bar{b}\bar{c}\bar{d}$</td>
<td>$a+b+c+d$</td>
</tr>
<tr>
<td>13</td>
<td>1101</td>
<td>$a\ b\ \bar{c}\ d$</td>
<td>?</td>
</tr>
<tr>
<td>15</td>
<td>1111</td>
<td>$a\ b\ c\ d$</td>
<td>$\bar{a}+\bar{b}+\bar{c}+\bar{d}$</td>
</tr>
</tbody>
</table>

Minterm and Maxterm Relationship

- **Review: DeMorgan's Theorem**
  \[ x \cdot y = \bar{x} + \bar{y} \quad \text{and} \quad x + y = \bar{x} \cdot \bar{y} \]
- **Two-variable example:**
  \[ M_2 = \bar{x} + y \quad \text{and} \quad m_2 = x \cdot \bar{y} \]
  Thus $M_2$ is the complement of $m_2$ and vice-versa.
- **Since DeMorgan's Theorem holds for $n$ variables,**
  the above holds for terms of $n$ variables
- **giving:**
  \[ M_i = \bar{m}_i \quad \text{and} \quad m_i = \bar{M}_i \]
  Thus $M_i$ is the complement of $m_i$. 
Function Tables for Both

- **Minterms of 2 variables**

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>m₀</th>
<th>m₁</th>
<th>m₂</th>
<th>m₃</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
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<tr>
<td>1</td>
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<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

- **Maxterms of 2 variables**

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>M₀</th>
<th>M₁</th>
<th>M₂</th>
<th>M₃</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
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<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Each column in the maxterm function table is the complement of the column in the minterm function table since Mᵢ is the complement of mᵢ.

**Observations**

- In the function tables:
  - Each minterm has one and only one 1 present in the 2ⁿ terms (a minimum of 1s). All other entries are 0.
  - Each maxterm has one and only one 0 present in the 2ⁿ terms. All other entries are 1 (a maximum of 1s).

- We can implement any function by "ORing" the minterms corresponding to "1" entries in the function table. These are called the minterms of the function.

- We can implement any function by "ANDing" the maxterms corresponding to "0" entries in the function table. These are called the maxterms of the function.

- This gives us two canonical forms:
  - Sum of Minterms (SOM)
  - Product of Maxterms (POM)

for stating any Boolean function.
Minterm Function Example

- Example: Find \( F_1 = m_1 + m_4 + m_7 \)

\[
F_1 = \overline{x} \overline{y} z + x \overline{y} \overline{z} + x y z
\]

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>z</th>
<th>index</th>
<th>( m_1 )</th>
<th>( m_4 )</th>
<th>( m_7 )</th>
<th>( F_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
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\[ F(A, B, C, D, E) = m_2 + m_9 + m_{17} + m_{23} \]

\[ F(A, B, C, D, E) = \]
Maxterm Function Example

- Example: Implement $F_1$ in maxterms:

$$F_1 = M_0 \cdot M_2 \cdot M_3 \cdot M_5 \cdot M_6$$

$$F_1 = (x + y + z) \cdot (x + \overline{y} + z) \cdot (x + \overline{y} + \overline{z})$$

$$\cdot (\overline{x} + y + \overline{z}) \cdot (\overline{x} + \overline{y} + z)$$

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</table>

Maxterm Function Example

- $F(A, B, C, D) = M_3 \cdot M_8 \cdot M_{11} \cdot M_{14}$
- $F(A, B, C, D) =$
Canonical Sum of Minterms

- Any Boolean function can be expressed as a Sum of Minterms.
  - For the function table, the minterms used are the terms corresponding to the 1’s.
  - For expressions, expand all terms first to explicitly list all minterms. Do this by “ANDing” any term missing a variable v with a term (v + v).
- Example: Implement \( f = x + \overline{x} \overline{y} \) as a sum of minterms.
  
  First expand terms: \( f = x(y + \overline{y}) + \overline{x} \overline{y} \)
  
  Then distribute terms: \( f = xy + x\overline{y} + \overline{x} \overline{y} \)
  
  Express as sum of minterms: \( f = m_3 + m_2 + m_0 \)

Another SOM Example

- Example: \( F = A + \overline{B} C \)
  - There are three variables, A, B, and C which we take to be the standard order.
  - Expanding the terms with missing variables:
    
    - Collect terms (removing all but one of duplicate terms):
    - Express as SOM:
Shorthand SOM Form

- From the previous example, we started with: 
  \[ F = A + B \overline{C} \]
- We ended up with: 
  \[ F = m_1+m_4+m_5+m_6+m_7 \]
- This can be denoted in the formal shorthand: 
  \[ F(A, B, C) = \Sigma_m(1, 4, 5, 6, 7) \]
- Note that we explicitly show the standard variables in order and drop the “m” designators.

Canonical Product of Maxterms

- Any Boolean Function can be expressed as a **Product of Maxterms (POM)**.
  - For the function table, the maxterms used are the terms corresponding to the 0's.
  - For an expression, expand all terms first to explicitly list all maxterms. Do this by first applying the second distributive law, “ORing” terms missing variable \( v \) with a term equal to \( \overline{v} \cdot \overline{v} \) and then applying the distributive law again.
- Example: Convert to product of maxterms:  
  \[ f(x, y, z) = x + \overline{x} \overline{y} \]
  Apply the distributive law:  
  \[ x + \overline{x} \overline{y} = (x + \overline{x})(x + \overline{y}) = 1 \cdot (x + \overline{y}) = x + \overline{y} \]
  Add missing variable \( z \):  
  \[ x + \overline{y} + z \cdot \overline{z} = (x + \overline{y} + z)(x + \overline{y} + \overline{z}) \]
  Express as POM:  
  \[ f = M_2 \cdot M_3 \]
Another POM Example

- Convert to Product of Maxterms:
  \[ f(A, B, C) = A \overline{C} + B C + \overline{A} \overline{B} \]
- Use \[ x + y \cdot z = (x+y) \cdot (x+z) \] with \[ x = (A \overline{C} + B C), \ y = \overline{A}, \] and \( z = \overline{B} \) to get:
  \[ f = (A \overline{C} + B C + \overline{A})(A \overline{C} + B C + \overline{B}) \]
- Then use \[ x + \overline{x} \cdot y = x + y \] to get:
  \[ f = (\overline{C} + B C + \overline{A})(A \overline{C} + C + \overline{B}) \]
- and a second time to get:
  \[ f = (\overline{C} + B + \overline{A})(A + C + \overline{B}) \]
- Rearrange to standard order,
  \[ f = (A + B + \overline{C})(A + \overline{B} + C) \] to give \( f = M_5 \cdot M_2 \)

Function Complements

- The complement of a function expressed as a sum of minterms is constructed by selecting the minterms missing in the sum-of-minterms canonical forms.
- Alternatively, the complement of a function expressed by a Sum of Minterms form is simply the Product of Maxterms with the same indices.
- Example: Given \( F(x, y, z) = \Sigma_m(1,3,5,7) \)
  \[ \overline{F}(x, y, z) = \Sigma_m(0,2,4,6) \]
  \( \overline{F}(x, y, z) = \Pi_m(1,3,5,7) \)
Conversion Between Forms

- To convert between sum-of-minterms and product-of-maxterms form (or vice-versa) we follow these steps:
  - Find the function complement by swapping terms in the list with terms not in the list.
  - Change from products to sums, or vice versa.
- Example: Given F as before: \( F(x, y, z) = \Sigma_m(1, 3, 5, 7) \)
- Form the Complement: \( \overline{F}(x, y, z) = \Sigma_m(0, 2, 4, 6) \)
- Then use the other form with the same indices – this forms the complement again, giving the other form of the original function: \( F(x, y, z) = \Pi_m(0, 2, 4, 6) \)

Standard Forms

- Standard Sum-of-Products (SOP) form: equations are written as an OR of AND terms
- Standard Product-of-Sums (POS) form: equations are written as an AND of OR terms
- Examples:
  - SOP: \( A \cdot B \cdot C + \overline{A} \cdot \overline{B} \cdot C + B \)
  - POS: \( (A + B) \cdot (A + \overline{B} + \overline{C}) \cdot C \)
- These “mixed” forms are neither SOP nor POS
  - \( (A \cdot B + C)(A + C) \)
  - \( A \cdot B \cdot C + A \cdot C(A + B) \)
Standard Sum-of-Products (SOP)

- A sum of minterms form for $n$ variables can be written down directly from a truth table.
  - Implementation of this form is a two-level network of gates such that:
    - The first level consists of $n$-input AND gates, and
    - The second level is a single OR gate (with fewer than $2^n$ inputs).
- This form often can be simplified so that the corresponding circuit is simpler.

Standard Sum-of-Products (SOP)

- A Simplification Example:
  - $F(A, B, C) = \Sigma m(1,4,5,6,7)$
  - Writing the minterm expression:
    $$ F = \overline{A} \overline{B} C + A \overline{B} \overline{C} + A \overline{B} C + A B \overline{C} + A B C $$
  - Simplifying:
    $$ F = $$

- Simplified $F$ contains 3 literals compared to 15 in minterm $F$
AND/OR Two-level Implementation of SOP Expression

- The two implementations for F are shown below – it is quite apparent which is simpler!

SOP and POS Observations

- The previous examples show that:
  - Canonical Forms (Sum-of-minterms, Product-of-Maxterms), or other standard forms (SOP, POS) differ in complexity
  - Boolean algebra can be used to manipulate equations into simpler forms.
  - Simpler equations lead to simpler two-level implementations

- Questions:
  - How can we attain a “simplest” expression?
  - Is there only one minimum cost circuit?
  - The next part will deal with these issues.
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