How do we draw the moment and shear diagram for an arbitrarily loaded beam?

Is there a easier and faster way to draw these diagrams rather than cutting the beam at specific point and finding the internal actions point by point?

**Side Note (Needed for Deflection Calculations): Shear and Moment Diagrams Using Area Method**
Shear and Moment Diagram Using the **Area Method**

**Positive Sign Convention for a Beam Member**

- **Pozitif yayılı dış kuvvet**
- **Pozitif iç kesme kuvveti**
- **Pozitif iç moment**
The beam is cut at distance $x$ along the longitudinal direction!

By applying static equilibrium equations, internal forces $V$ and $M$ can be expressed in terms of external distributed force $w$. 

The diagrams illustrate the forces and moments acting on the beam at section $A$. The expressions for $V$ and $M$ are given as:

- Vertical force $V$: 
  \[ V = \frac{wL}{2} \]

- Moment $M$: 
  \[ M = wL \(x - \frac{x}{2}) = \frac{wx}{2} \]
By applying equations of equilibrium:

\[ V(x) = \frac{wL}{2} - wx \]
\[ M(x) = -\frac{wx^2}{2} + \frac{wL}{2}x \]
Consider the arbitrarily loaded beam shown below. We want to draw the necessary diagrams using the area method. 

In order to do this we need to develop mathematical relations between the external loads and internal actions, namely shear force and moment. To do this, consider a beam segment of length $\Delta x$. 

Shear and Moment Diagram Using the Area Method
Shear and Moment Diagram Using the Area Method

The plot given on the side is the free body diagram of the segment with length $\Delta x$. Notice that the distributed load is shown as an equivalent concentrated force.

Equilibrium of this segment gives the following:

$$
\sum F_y = 0; \quad V + w(x)\Delta x - (V + \Delta V) = 0
$$

$$
\Delta V = w(x)\Delta x \quad (1)
$$

$$
\sum M_O = 0; \quad -V\Delta x - M - w(x)\Delta x[k\Delta x] + (M + \Delta M) = 0
$$

$$
\Delta M = V\Delta x + w(x)k\Delta x^2 \quad (2)
$$

If we divide (1) and (2) with $\Delta x$ and calculate the limits at $\Delta x$ goes to zero, the following very important expressions will be obtained:

$$
\frac{dV}{dx} = w(x) \quad (3)
$$

$$
\frac{dM}{dx} = V \quad (4)
$$
Shear and Moment Diagram Using the **Area Method**

These two expressions will be used to construct the shear and moment diagrams:

\[ \frac{dV}{dx} = w(x) \]

\[ \frac{dM}{dx} = V \]

\[ w = \text{negative increasing slope} \]

\[ V = \text{Positive decreasing slope} \]
Shear and Moment Diagram Using the **Area Method**

Equations (3) and (4) can be written as follows:

\[
\frac{dV}{dx} = w(x) \quad (3)
\]

\[
\frac{dM}{dx} = V \quad (4)
\]

\[dV = w(x)dx\]

\[dM = V(x)dx\]

If we integrate both sides we obtain another useful set of equations:

\[\Delta V = \int w(x)dx\]

\[\Delta M = \int V(x)dx\]
Shear and Moment Diagram Using the **Area Method**

\[ \Delta V = \int w(x) \, dx \]

\[ \Delta M = \int V(x) \, dx \]
How do we deal with the concentrated load and concentrated moment in the area method? To do that consider the following drawings

\[ \Delta V = -F \]
If force F is directed downward, the change in the shear diagram will be a jump along the direction of the force F, and vice versa.

\[ \Delta M = M_0 \]
If \( M_0 \) is acting in clock wise direction, the change in the moment diagram will be a jump in the positive direction, and vice versa.
Example 1: Draw the shear and moment diagram of the cantilever beam shown below using the area method.

First let’s find the reaction forces (apply EoE):
Example 1 (cont.'ed):

Recall the expressions developed for the area method:

\[
\frac{dV}{dx} = w(x) \quad \Delta V = \int w(x) \, dx \\
\frac{dM}{dx} = V \quad \Delta M = \int V(x) \, dx
\]

\[
\Delta V = -F \downarrow - \\
\Delta M = M_0 \leftarrow +
\]

Directed downwards, Jump along the direction of force P

\( w = 0 \)  
Slope = 0

Since there is no distributed force, the slope of the shear diagram is zero.
Example 2: Draw the shear force and moment diagrams for the simply supported beam shown below:

First find the support reactions:

Apply EoE:
\[
\begin{align*}
\sum F_x &= 0 \\
\sum F_y &= 0 \\
\sum M &= 0
\end{align*}
\]
Example 2 (cont.’ed):

Recall the expressions developed for the area method:

\[
\frac{dV}{dx} = w(x)
\]

\[
\frac{dM}{dx} = V
\]

\[
\Delta V = \int w(x) \, dx
\]

\[
\Delta M = \int V(x) \, dx
\]

\[
\Delta V = -F
\]

\[
\Delta M = M_0
\]

Mo is in the clock wise direction (+), this leads to a positive jump in the moment diagram.

Slope = constant negative

w = 0
slope = 0
**Example 3:** Draw the shear force and moment diagrams for the cantilever beam shown below:

First find the support reactions by applying EoE:

\[
\sum F_x = 0 \\
\sum F_y = 0 \\
\sum M = 0
\]
Example 3 (cont.’ed):

Use the expressions derived for the area method. Notice that this time we have distributed force on the beam.

\[
\frac{dV}{dx} = w(x)
\]

\[
\frac{dM}{dx} = V
\]

\[
\Delta V = \int w(x) \, dx
\]

\[
\Delta M = \int V(x) \, dx
\]

\( w = \text{negative constant } (-w_0) \)

Slope = negatif constant \(-w_0\)

\( V = \text{Positive decreasing} \)

Slope = positive decreasing
Homework: Draw the shear force and moment diagrams for the cantilever beam shown below. Notice that we have concentrated as well as distributed external loads.
Chapter Objectives

- Determine the deflection and slope at specific points on beams and shafts, using various analytical methods including:
  - The integration method

- Determine the same, using a semi-graphical technique, called the moment-area method (*if time permits*).
1) The slope angle $\theta$ in flexure equations is

a) Measured in degree

b) Measured in radian

c) Exactly equal to $dv/dx$

d) None of the above
2) The load must be limited to a magnitude so that not to change significantly the original geometry of the beam. This is the assumption for

a) The method of superposition

b) The moment area method

c) The method of integration

d) All of them
APPLICATIONS (UPDATE!)

The Royal Gorge Bridge
Colorado
The Forth Road Bridge
Scotland
APPLICATIONS (UPDATE!)

The Golden Gate Bridge
San Francisco
Seismic performance of well-confined concrete bridge

Deformed reinforcing bars
Description: **Strong column-weak girder** assemblage of a ductile moment-resistant reinforced concrete frame. In the experiments conducted on subassemblages of ductile reinforced concrete moment-resistant frames it has been observed that there was significant degradation in stiffness and strength of frames with repeated cycles of deformation reversal.

Experiments conducted at UC Berkeley on subassemblages in which the plastic hinges have been moved away from the columns, as illustrated in this slide, and therefore keeping the joint elastic, have shown that it is possible to achieve good stable **hysteretic behavior**. Note in this slide that all the **inelastic deformations** occurred in the beams in regions away from the face of the column.
ELASTIC CURVE

• The deflection diagram of the longitudinal axis that passes through the centroid of each cross-sectional area of the beam is called the elastic curve, which is characterized by the deflection and slope along the curve.
ELASTIC CURVE (cont)

- Moment-curvature relationship:
  - Sign convention:
ELASTIC CURVE (cont)

- Consider a segment of width \( dx \), the strain in area \( ds \), located at a position \( y \) from the neutral axis is \( \varepsilon = \frac{ds' - ds}{ds} \). However, \( ds = dx = \rho d\theta \) and \( ds' = (\rho - y) d\theta \), and so \( \varepsilon = \frac{[(\rho - y) d\theta - \rho d\theta]}{(\rho d\theta)} \)

\[
\frac{1}{\rho} = -\frac{\varepsilon}{y}
\]

- Comparing with the Hooke’s Law \( \varepsilon = \sigma / E \) and the flexure formula \( \sigma = -My/l \)

\[
\frac{1}{\rho} = \frac{M}{EI} \quad \text{or} \quad \frac{1}{\rho} = -\frac{\sigma}{Ey}
\]
SLOPE AND DISPLACEMENT BY INTEGRATION

- Kinematic relationship between radius of curvature $\rho$ and location $x$:

$$\frac{1}{\rho} = -\frac{d^2v/dv^2}{\left[1 + (dv/dx)^2\right]^{3/2}}$$

- Then using the moment curvature equation, we have

$$\frac{M}{EI} = \frac{1}{\rho} = \frac{d^2v/dx^2}{\left[1 + (dv/dx)^2\right]^{3/2}} \approx \frac{d^2v}{dx^2}$$
SLOPE AND DISPLACEMENT BY INTEGRATION (cont)

- Sign convention:
Boundary Conditions:

- The integration constants can be determined by imposing the boundary conditions, or

- Continuity condition at specific locations
EXAMPLE 1

The cantilevered beam shown in Fig. 12–10a is subjected to a vertical load $P$ at its end. Determine the equation of the elastic curve. $EI$ is constant.
EXAMPLE 1 (cont)

Solutions

• From the free-body diagram, with $M$ acting in the positive direction, Fig. 12–10b, we have

$$M = -Px$$

• Applying Eq. 12–10 and integrating twice yields

$$EI \frac{d^2 v}{dx^2} = -Px \quad (1)$$

$$EI \frac{dv}{dx} = -\frac{Px^2}{2} + C_1 \quad (2)$$

$$EI v = -\frac{Px^3}{6} + C_1 x + C_2 \quad (1)$$
Solutions

- Using the boundary conditions \( \frac{dv}{dx} = 0 \) at \( x = L \) and \( v = 0 \) at \( x = L \), equations 2 and 3 become

\[
0 = -\frac{PL^2}{2} + C_1
\]

\[
0 = -\frac{PL^3}{6} + C_1L + C_2
\]

\[
\Rightarrow C_1 = \frac{PL^2}{2} \quad \text{and} \quad C_2 = -\frac{PL^3}{3}
\]

- Substituting these results, we get

\[
\theta = \frac{P}{2EI} \left( L^2 - x^2 \right)
\]

\[
v = \frac{P}{6EI} \left( -x^3 + 3L^2x - 2L^3 \right) \quad \text{(Ans)}
\]
• Maximum slope and displacement occur at for which $A(x = 0)$,

\[ \theta_A = \frac{PL^2}{2EI} \quad (4) \]

\[ \nu_A = -\frac{PL^3}{3EI} \quad (5) \]

• If this beam was designed without a factor of safety by assuming the allowable normal stress is equal to the yield stress is 250 MPa; then a W310 x 39 would be found to be adequate ($I = 84.4(10^6)\text{mm}^4$)

\[ \theta_A = \frac{30(5)^2(1000)^2}{2[200][84.4(10^6)]} = 0.0222 \text{ rad} \]

\[ \nu_A = -\frac{30(5)^2(1000)^2}{3[200][84.4(10^6)]} = -74.1 \text{ mm} \]
The simply supported beam shown in Fig. 12–12a is subjected to the concentrated force. Determine the maximum deflection of the beam. $EI$ is constant.
EXAMPLE 12.3 CONTINUED

SOLUTION

Elastic Curve. The beam deflects as shown in Fig. 12–12b. Two coordinates must be used, since the moment function will change at B. Here we will take \( x_1 \) and \( x_2 \), having the same origin at A.

Moment Function. From the free-body diagrams shown in Fig. 12–12c,

\[
M_1 = 2x_1 \\
M_2 = 2x_2 - 6(x_2 - 2) = 4(3 - x_2)
\]

Slope and Elastic Curve. Applying Eq. 12–10 for \( M_1 \), for \( 0 \leq x_1 < 2 \) m, and integrating twice yields
EXAMPLE 12.3 CONTINUED

\[ EI \frac{d^2v_1}{dx_1^2} = 2x_1 \]

\[ EI \frac{dv_1}{dx_1} = x_1^2 + C_1 \]  \quad (1)

\[ EIv_1 = \frac{1}{3}x_1^3 + C_1x_1 + C_2 \]  \quad (2)

Likewise for \( M_2 \), for \( 2 \text{ m} < x_2 \leq 3 \text{ m} \),

\[ EI \frac{d^2v_2}{dx_2^2} = 4(3 - x_2) \]

\[ EI \frac{dv_2}{dx_2} = 4\left(3x_2 - \frac{x_2^2}{2}\right) + C_3 \]  \quad (3)

\[ EIv_2 = 4\left(\frac{3}{2}x_2^2 - \frac{x_2^3}{6}\right) + C_3x_2 + C_4 \]  \quad (4)

(c)

Fig. 12–12
The four constants are evaluated using two boundary conditions, namely, \( x_1 = 0, \ v_1 = 0 \) and \( x_2 = 3 \, \text{m}, \ v_2 = 0 \). Also, two continuity conditions must be applied at \( B \), that is, \( \frac{dv_1}{dx_1} = \frac{dv_2}{dx_2} \) at \( x_1 = x_2 = 2 \, \text{m} \) and \( v_1 = v_2 \) at \( x_1 = x_2 = 2 \, \text{m} \). Substitution as specified results in the following four equations:

\[
\begin{align*}
v_1 &= 0 \text{ at } x_1 = 0; \quad 0 = 0 + 0 + C_2 \\
v_2 &= 0 \text{ at } x_2 = 3 \, \text{m}; \quad 0 = 4 \left( \frac{3}{2} (3)^2 - \frac{(3)^3}{6} \right) + C_3(3) + C_4 \\
\frac{dv_1}{dx_1} \bigg|_{x=2 \, \text{m}} &= \frac{dv_2}{dx_2} \bigg|_{x=2 \, \text{m}} \quad ; \quad (2)^2 + C_1 = 4 \left( 3(2) - \frac{(2)^3}{2} \right) + C_3 \\
v_1(2 \, \text{m}) &= v_2(2 \, \text{m}); \quad \frac{1}{3} (2)^3 + C_1(2) + C_2 = 4 \left( \frac{3}{2} (2)^2 - \frac{(2)^3}{6} \right) + C_3(2) + C_4
\end{align*}
\]

Solving, we get

\[
\begin{align*}
C_1 &= \frac{8}{3} \quad & C_2 &= 0 \\
C_3 &= -\frac{44}{3} \quad & C_4 &= 8
\end{align*}
\]
EXAMPLE 12.3 CONTINUED

Thus Eqs. 1–4 become

\[ EI \frac{dv_1}{dx_1} = x_1^2 - \frac{8}{3} \]  \hspace{1cm} (5)

\[ EI v_1 = \frac{1}{3} x_1^3 - \frac{8}{3} x_1 \]  \hspace{1cm} (6)

\[ EI \frac{dv_2}{dx_2} = 12 x_2 - 2 x_2^2 - \frac{44}{3} \]  \hspace{1cm} (7)

\[ EI v_2 = 6 x_2^2 - \frac{2}{3} x_2^3 - \frac{44}{3} x_2 + 8 \]  \hspace{1cm} (8)

By inspection of the elastic curve, Fig. 12–12b, the maximum deflection occurs at D, somewhere within region AB. Here the slope must be zero. From Eq. 5,

\[ x_1^2 - \frac{8}{3} = 0 \]

\[ x_1 = 1.633 \]

Substituting into Eq. 6,

\[ v_{\text{max}} = \frac{-2.90 \text{ kN} \cdot \text{m}^3}{EI} \]  \hspace{1cm} \text{Ans.}

The negative sign indicates that the deflection is downward.