A partial differential equation problem is said to be well-posed if
- a solution to the problem exist.
- the solution is unique.
- the solution depends continuously on the problem data (stability).

If one of these conditions is not satisfied, the partial differential equation is said to be ill-posed.

In practice, the question whether a PDE problem is well-posed can be difficult to settle. Roughly speaking the following guidelines apply:
- The auxiliary conditions imposed must not be too many or a solution will not exist.
- The auxiliary conditions imposed must not be too few or the solution will not be unique.
- The kind of auxiliary conditions must be correctly matched to the type of the PDE or the solution will not depend continuously on data.
Example: The boundary value problem (BVP)

\[ u''(x) = 0 \quad (0 < x < 1), \quad u'(0) = 0, \quad u'(1) = 1 \]

has no solutions. The problem is overdetermined.

Example: The boundary value problem (BVP)

\[ u''(x) = 0 \quad (0 < x < 1), \quad u'(0) = 0, \quad u'(1) = 0 \]

has infinitely many solutions, namely solutions of the form \( u = \text{constant} \). The problem is underdetermined.

Example: The two dimensional Laplace equation

\[ u_{xx} + u_{yy} = 0 \]

on the domain \( y > 0 \) with boundary conditions \( u(x, 0) = 0 \) and \( u_y(x, 0) = 0 \) has the solution \( u \equiv 0 \).

If the second boundary condition is changed to \( u_y(x, 0) = e^{-\sqrt{n}x} \sin nx \), the solution becomes

\[ u(x, y) = \frac{1}{n} e^{-\sqrt{n}x} \sin nx \sinh(ny) \]

We can choose \( n \) large enough to make \( \max_x |u_y(x, 0)| \) as small as we like, but no matter how small the perturbation, the solution always blows up as \( y \to \infty \). Thus, the problem is unstable: the solution does not depend continuously on the boundary data.
Let us consider the following problem

\[ Au = f, \quad u \in U, \quad f \in F \]

where \( A : U \to F \) is a linear operator. \( U \) and \( F \) are linear normed spaces.

The problem is called well-posed on the class of its admissible data if the solution:

- **exist**: \( \forall f \in F \; \exists u \in U \) such that \( Au = f \).
- **unique**: \( \forall f \in F \; \exists \tilde{u} \in U \) such that \( Au = f \), \( A \tilde{u} = \tilde{f} \) then \( u = \tilde{u} \).
- **is stable**: \( \forall f \in F \; \forall \varepsilon > 0 \; \exists \delta(\epsilon, f) > 0 \) such that for any element \( \tilde{f} \in F \) \( \forall u, \tilde{u} \in U \) satisfying \( Au = f \), \( A \tilde{u} = \tilde{f} \) if \( \rho_F(f, \tilde{f}) < \delta \) then \( \rho_U(u, \tilde{u}) < \varepsilon \).

The problem is called ill-posed problem if one of these conditions does not hold.

**Differentiation Problem:**

\[ u = f'(x) \Leftrightarrow Au = f, \quad Au = \int_0^x u(\xi)\,d\xi, \quad u \in C(\mathbb{R}) = U \]

\[ \rho_1(f, \tilde{f}) = \sup_{x \in \mathbb{R}} |f(x) - \tilde{f}(x)| + |f'(x) - \tilde{f}'(x)| \]

\[ \rho_0(u, \tilde{u}) = \sup_{x \in \mathbb{R}} |u(x) - \tilde{u}(x)| \]

1. Differentiation problem is well-posed on \((U_0, F_1)\).
2. Differentiation problem is ill-posed on \((U_0, F_0)\).
1) \[ \tilde{u} = \tilde{f}'(x), \quad u = f'(x) \]

\[ |u(x) - \tilde{u}(x)| = |f'(x) - \tilde{f}'(x)| \leq |f(x) - \tilde{f}(x)| + |f'(x) - \tilde{f}'(x)| \leq \rho_1(f, \tilde{f}) \]

\[ \forall x \in \mathbb{R}, \quad \rho_0(u, \tilde{u}) \leq \rho_1(f, \tilde{f}). \]

2) Let \( f(x) \in C^1(\mathbb{R}) \); \[ \tilde{f}(x) = f(x) + \frac{1}{n} \sin(n^2x) \]

\[ \rho_0(f, \tilde{f}) = \sup_{x \in \mathbb{R}} \left| \frac{1}{n} \sin(n^2x) \right| = \frac{1}{n} \]

\[ \rho_0(u, \tilde{u}) = \sup_{x \in \mathbb{R}} \left| n \cos(n^2x) \right| = n \]

\[ \forall \delta > 0 \ \exists n : \frac{1}{n} < \delta \text{ and } n > 1 \quad \rho_0(f, \tilde{f}) < \delta \text{ but } \rho_0(u, \tilde{u}) > 1. \]

Therefore the problem is ill posed.
The Cauchy Problem for the Laplace Equation

\[ \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0, \quad x \in [0, X], \quad y \in \mathbb{R} \]

\[ U(0, y) = \varphi(y), \quad \frac{\partial U}{\partial x}(x, y) \bigg|_{x=0} = \psi(y) = 0 \]

where \( X \) is a given positive number. \( \varphi(y) \) and \( \psi(y) \) are given smooth functions.

Remark:

\[ \mathcal{U}_l = (C[0, X] \times \mathbb{R}, \rho_l) \]

\[ \rho_l(U, \tilde{U}) = \max_{x \in [0, X]} \sup_{y \in \mathbb{R}} \left\{ \sum_{|\alpha| \leq l} |D^\alpha[U(x, y) - \tilde{U}(x, y)]| \right\} \]

\[ \mathcal{F}_m = (C^m(\mathbb{R}), \rho_m) \]

\[ \rho_m(\varphi, \tilde{\varphi}) = \sup_{y \in \mathbb{R}} \left\{ \sum_{j=0}^{m} \left| \frac{d^j[\varphi(y) - \tilde{\varphi}(y)]}{dy^j} \right| \right\} \]

Here \( D^\alpha u \) is an abbreviated form of \( \frac{\partial^{\alpha_1} U(x, y)}{\partial x^{\alpha_1} \partial y^{\alpha_2}} \)
Theorem: The Cauchy’s problem for the Laplace equation is ill-posed problem on every pair spaces \((U_l, F_m)\).

Proof:

Lemma 1: If \(\varphi_k(y) = \frac{1}{k^s} \sin ky\) and \(\psi_k = 0\) then
\[
U_k(x, y) = \frac{\cosh(kx)}{k^s} \sin ky
\]
satisfies the Cauchy problem.

Lemma 2: If \(\varphi_k(y) = \frac{1}{k^s} \sin ky\) then \(\rho_m(\varphi_k, \varphi) \leq \frac{2}{k^{s-m}}\) where \(s - m > 0\) and \(\rho_m(\varphi_k, \varphi) \to 0\) as \(k \to \infty\).

Lemma 3: If \(U_k(x, y) = \frac{1}{k^s} \sin ky\) then there exists \(k_0\) such that for \(k \geq k_0\), \(\rho_l(U_k, U) > k\) for \(l \geq 0\) and \(U = 0\). Therefore \(\rho_l(U_k, U) \to \infty\) as \(k \to \infty\), \(s - m > 0\).

Backward's Diffusion Equation Problem
Consider the following diffusion equation:

\[
\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}, \quad x \in (0, \pi), \ t > 0
\]
\[
U(0, t) = U(\pi, t), \quad t > 0
\]
\[
U(x, T) = \psi(x), \quad x \in (0, \pi)
\]

Let \(T > 0\) be a fixed given number. \(\psi(x)\) be a given smooth function. We need to find a function \(U(x, 0) = \varphi(x)\) if the function \(U(x, y)\) satisfies the diffusion equation and the conditions.
Well-posed and Ill-posed Problems

Remark:

\[ U_l = (C^l, \rho_l); \varphi(x) \in C^l, 0 \leq x \leq \pi \]

\[ \rho_l(\varphi, \tilde{\varphi}) = \max_{x \in [0, \pi]} \left\{ |\varphi(x) - \tilde{\varphi}(x)| + \sum_{j=1}^{l} \frac{d_j^l[\varphi(x) - \tilde{\varphi}(x)]}{dx} \right\} \]

\[ F_m = (C^m, \rho_m); \psi(x) \in C^m(\mathbb{R}) \]

\[ \rho_m(\psi, \tilde{\psi}) = \max_{x \in [0, \pi]} \left\{ |\psi(x) - \tilde{\psi}(x)| + \sum_{j=1}^{m} \frac{d_j^m[\psi(x) - \tilde{\psi}(x)]}{dx} \right\} \]

Theorem: Backward's diffusion equation problem is ill-posed problem on every pair spaces \((U_l, F_m)\).

Proof:

Lemma 1: If \( \varphi_n(x) = e^{n^2 T \sin nx} \) and \( \psi_n = \frac{\sin nx}{n^s} \) then

\[ U_n(x, t) = e^{n^2 (T-t) \frac{\sin nx}{n^s}} \] satisfies the Cauchy problem.

Lemma 2: If \( \psi_n(x) = \frac{\sin nx}{n^s} \) then \( \rho_m(\psi_n, \psi) \leq \frac{2}{n^{s-m}} \) where \( s - m > 0 \) and \( \rho_m(\psi_n, \psi) \to 0 \) as \( n \to \infty \).

Lemma 3: If \( \varphi_n = e^{n^2 T \sin nx} \) then there exists \( n_0 \) such that for \( n \geq n_0, \rho_l(\varphi_n, \varphi) > n \) for \( l \geq 0 \) where \( \varphi = 0 \). Therefore \( \rho_l(U_n, U) \to \infty \) as \( n \to \infty \), \( s - m > 0 \).