Distributed Forces: Centroids and Centers of Gravity
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Introduction

• The earth exerts a gravitational force on each of the particles forming a body. These forces can be replace by a single equivalent force equal to the weight of the body and applied at the center of gravity for the body.

• The centroid of an area is analogous to the center of gravity of a body. The concept of the first moment of an area is used to locate the centroid.

• Determination of the area of a surface of revolution and the volume of a body of revolution are accomplished with the Theorems of Pappus-Guldinus.
Center of Gravity of a 2D Body

- Center of gravity of a plate

\[
\sum M_y \quad xW = \sum x\Delta W \\
= \int x \, dW \\
\sum M_y \quad yW = \sum y\Delta W \\
= \int y \, dW
\]

- Center of gravity of a wire
Centroids and First Moments of Areas and Lines

- **Centroid of an area**

  \[ \bar{x}W = \int x\,dW \]

  \[ \bar{x}(\gamma A_t) = \int x(\gamma t)\,dA \]

  \[ \bar{x}A = \int x\,dA = Q_y \]

  \[ \bar{y}A = \int y\,dA = Q_x \]

  = first moment with respect to \( y \)

- **Centroid of a line**

  \[ \bar{x}W = \int x\,dW \]

  \[ \bar{x}(\gamma L) = \int x(\gamma L)\,dL \]

  \[ \bar{x}L = \int x\,dL \]

  \[ \bar{y}L = \int y\,dL \]

  = first moment with respect to \( x \)
First Moments of Areas and Lines

- An area is symmetric with respect to an axis \(BB'\) if for every point \(P\) there exists a point \(P'\) such that \(PP'\) is perpendicular to \(BB'\) and is divided into two equal parts by \(BB'\).
- The first moment of an area with respect to a line of symmetry is zero.
- If an area possesses a line of symmetry, its centroid lies on that axis.
- If an area possesses two lines of symmetry, its centroid lies at their intersection.
- An area is symmetric with respect to a center \(O\) if for every element \(dA\) at \((x,y)\) there exists an area \(dA'\) of equal area at \((-x,-y)\).
- The centroid of the area coincides with the center of symmetry.
# Centroids of Common Shapes of Areas

<table>
<thead>
<tr>
<th>Shape</th>
<th>( \bar{x} )</th>
<th>( \bar{y} )</th>
<th>Area</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triangular area</td>
<td></td>
<td>( \frac{h}{2} )</td>
<td>( \frac{bh}{2} )</td>
</tr>
<tr>
<td>Quarter-circular area</td>
<td></td>
<td>( \frac{4r}{3\pi} )</td>
<td>( \frac{\pi r^2}{4} )</td>
</tr>
<tr>
<td>Semicircular area</td>
<td>0</td>
<td>( \frac{4r}{3\pi} )</td>
<td>( \frac{\pi r^2}{2} )</td>
</tr>
<tr>
<td>Quarter-elliptical area</td>
<td></td>
<td>( \frac{4a}{3\pi} )</td>
<td>( \frac{\pi ab}{4} )</td>
</tr>
<tr>
<td>Semielliptical area</td>
<td>0</td>
<td>( \frac{4b}{3\pi} )</td>
<td>( \frac{\pi ab}{2} )</td>
</tr>
<tr>
<td>Semiparabolic area</td>
<td>( \frac{3a}{8} )</td>
<td>( \frac{3h}{5} )</td>
<td>( \frac{2ah}{3} )</td>
</tr>
<tr>
<td>Parabolic area</td>
<td>0</td>
<td>( \frac{3h}{5} )</td>
<td>( \frac{4ah}{3} )</td>
</tr>
<tr>
<td>Parabolic spandrel</td>
<td>( \frac{3a}{4} )</td>
<td>( \frac{3h}{10} )</td>
<td>( \frac{ah}{3} )</td>
</tr>
<tr>
<td>General spandrel</td>
<td>( \frac{n+1}{n+2}a )</td>
<td>( \frac{n+1}{4n+2}h )</td>
<td>( \frac{ah}{n+1} )</td>
</tr>
<tr>
<td>Circular sector</td>
<td></td>
<td>( \frac{2r \sin \alpha}{3 \alpha} )</td>
<td>( \frac{\pi r^2}{3} )</td>
</tr>
</tbody>
</table>
Centroids of Common Shapes of Lines

<table>
<thead>
<tr>
<th>Shape</th>
<th>$\bar{x}$</th>
<th>$\bar{y}$</th>
<th>Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quarter-circular arc</td>
<td>$\frac{2r}{\pi}$</td>
<td>$\frac{2r}{\pi}$</td>
<td>$\frac{\pi r}{2}$</td>
</tr>
<tr>
<td>Semicircular arc</td>
<td>0</td>
<td>$\frac{2r}{\pi}$</td>
<td>$\pi r$</td>
</tr>
<tr>
<td>Arc of circle</td>
<td>$\frac{r \sin \alpha}{\alpha}$</td>
<td>0</td>
<td>$2\alpha r$</td>
</tr>
</tbody>
</table>
Composite Plates and Areas

• Composite plates

\[
\bar{X} \sum W = \sum \bar{x} W \\
\bar{Y} \sum W = \sum \bar{y} W
\]

• Composite area

\[
\bar{X} \sum A = \sum \bar{x} A \\
\bar{Y} \sum A = \sum \bar{y} A
\]
Sample Problem 5.1

For the plane area shown, determine the first moments with respect to the $x$ and $y$ axes and the location of the centroid.

SOLUTION:

- Divide the area into a triangle, rectangle, and semicircle with a circular cutout.
- Calculate the first moments of each area with respect to the axes.
- Find the total area and first moments of the triangle, rectangle, and semicircle. Subtract the area and first moment of the circular cutout.
- Compute the coordinates of the area centroid by dividing the first moments by the total area.
Sample Problem 5.1

Find the total area and first moments of the triangle, rectangle, and semicircle. Subtract the area and first moment of the circular cutout.

\[
\begin{align*}
\Sigma A &= 13.828 \times 10^3 \\
\Sigma \bar{x}A &= +757.7 \times 10^3 \\
\Sigma \bar{y}A &= +506.2 \times 10^3
\end{align*}
\]

\[
\begin{align*}
Q_x &= +506.2 \times 10^3 \text{ mm}^3 \\
Q_y &= +757.7 \times 10^3 \text{ mm}^3
\end{align*}
\]
Sample Problem 5.1

- Compute the coordinates of the area centroid by dividing the first moments by the total area.

\[ \bar{X} = \frac{\sum xA}{\sum A} = \frac{+757.7 \times 10^3 \, \text{mm}^3}{13.828 \times 10^3 \, \text{mm}^2} \]

\[ \bar{X} = 54.8 \, \text{mm} \]

\[ \bar{Y} = \frac{\sum yA}{\sum A} = \frac{+506.2 \times 10^3 \, \text{mm}^3}{13.828 \times 10^3 \, \text{mm}^2} \]

\[ \bar{Y} = 36.6 \, \text{mm} \]
Determination of Centroids by Integration

\[
\bar{x}A = \int x
dA = \iint x
dx
dy = \int \bar{x}_e dA
\]

\[
\bar{y}A = \int y
dA = \iint y
dx
dy = \int \bar{y}_e dA
\]

\[
\bar{x}A = \int \bar{x}_e
dA
= \int x \left( y
dx \right)
\]

\[
\bar{y}A = \int \bar{y}_e
dA
= \int \frac{y}{2} \left( y
dx \right)
\]

- Double integration to find the first moment may be avoided by defining \( dA \) as a thin rectangle or strip.
Sample Problem 5.4

Determine by direct integration the location of the centroid of a parabolic spandrel.

\[ y = kx^2 \]

SOLUTION:
- Determine the constant \( k \).
- Evaluate the total area.
- Using either vertical or horizontal strips, perform a single integration to find the first moments.
- Evaluate the centroid coordinates.
SOLUTION:

• Determine the constant $k$.

$$y = k x^2$$

$$b = k a^2 \quad \Rightarrow \quad k = \frac{b}{a^2}$$

$$y = \frac{b}{a^2} x^2 \quad \text{or} \quad x = \frac{a}{b^{1/2}} y^{1/2}$$

• Evaluate the total area.

$$A = \int dA$$

$$= \int y \, dx = \int_{a}^{b} \frac{b}{a^2} x^2 \, dx = \left[ \frac{b}{a^2} \frac{x^3}{3} \right]_{a}^{b}$$

$$= \frac{ab}{3}$$
Sample Problem 5.4

- Using vertical strips, perform a single integration to find the first moments.

\[
Q_y = \int \bar{x}_{el} dA = \int xy\,dx = \int_0^a x \left( \frac{b}{a^2} x^2 \right) dx
\]

\[
= \left[ \frac{b}{a^2} \frac{x^4}{4} \right]_0^a = \frac{a^2b}{4}
\]

\[
Q_x = \int \bar{y}_{el} dA = \int \frac{y}{2} y\,dx = \int_0^a \frac{1}{2} \left( \frac{b}{a^2} x^2 \right)^2 dx
\]

\[
= \left[ \frac{b^2}{2a^4} \frac{x^5}{5} \right]_0^a = \frac{ab^2}{10}
\]
Sample Problem 5.4

- Or, using horizontal strips, perform a single integration to find the first moments.

\[ Q_y = \int \bar{x}_{el} \, dA = \int_a^b \frac{a + x}{2} (a - x) \, dy = \int_0^b \frac{a^2 - x^2}{2} \, dy \]

\[ = \frac{1}{2} \int_0^b \left( a^2 - \frac{a^2}{b} y \right) \, dy = \frac{a^2 b}{4} \]

\[ Q_x = \int \bar{y}_{el} \, dA = \int y (a - x) \, dy = \int y \left( a - \frac{a}{b^{1/2}} y^{1/2} \right) \, dy \]

\[ = \int_0^b \left( ay - \frac{a}{b^{1/2}} y^{3/2} \right) \, dy = \frac{ab^2}{10} \]
Sample Problem 5.4

- Evaluate the centroid coordinates.

\[
\bar{x} A = Q_y
\]
\[
\bar{x} \frac{ab}{3} = \frac{a^2b}{4}
\]
\[
\bar{x} = \frac{3}{4} a
\]

\[
\bar{y} A = Q_x
\]
\[
\bar{y} \frac{ab}{3} = \frac{ab^2}{10}
\]
\[
\bar{y} = \frac{3}{10} b
\]
Theorems of Pappus-Guldinus

- Surface of revolution is generated by rotating a plane curve about a fixed axis.

- Area of a surface of revolution is equal to the length of the generating curve times the distance traveled by the centroid through the rotation.

\[ A = 2\pi \bar{y}L \]
Theorems of Pappus-Guldinus

- Body of revolution is generated by rotating a plane area about a fixed axis.

\[ V = 2\pi \bar{y} A \]
The outside diameter of a pulley is 0.8 m, and the cross section of its rim is as shown. Knowing that the pulley is made of steel and that the density of steel is determined, the mass and weight of the rim.

SOLUTION:

• Apply the theorem of Pappus-Guldinus to evaluate the volumes or revolution for the rectangular rim section and the inner cutout section.

• Multiply by density and acceleration to get the mass and acceleration.
Sample Problem 5.7

SOLUTION:

- Apply the theorem of Pappus-Guldinus to evaluate the volumes of revolution for the rectangular rim section and the inner cutout section.

- Multiply by density and acceleration to get the mass and acceleration.

<table>
<thead>
<tr>
<th>Area, mm²</th>
<th>( \bar{y} ), mm</th>
<th>Distance Traveled by ( C ), mm</th>
<th>Volume, mm³</th>
</tr>
</thead>
<tbody>
<tr>
<td>I +5000</td>
<td>375</td>
<td>( 2\pi(375) = 2356 )</td>
<td>((5000)(2356) = 11.78 \times 10^6)</td>
</tr>
<tr>
<td>II -1800</td>
<td>365</td>
<td>( 2\pi(365) = 2293 )</td>
<td>((-1800)(2293) = -4.13 \times 10^6)</td>
</tr>
</tbody>
</table>

Volume of rim = \( 7.65 \times 10^6 \) mm³

\[ m = \rho V = \left(7.85 \times 10^3 \text{ kg/m}^3\right)\left(7.65 \times 10^6 \text{ mm}^3\right)\left(10^{-9} \text{ m}^3/\text{mm}^3\right) = 60.0 \text{ kg} \]

\[ W = mg = (60.0 \text{ kg})(9.81 \text{ m/s}^2) = 589 \text{ N} \]
Distributed Loads on Beams

- A distributed load is represented by plotting the load per unit length, \( w \) (N/m). The total load is equal to the area under the load curve.

\[
W = \int_{0}^{L} w \, dx = \int dA = A
\]

- A distributed load can be replaced by a concentrated load with a magnitude equal to the area under the load curve and a line of action passing through the area centroid.

\[
(OP)W = \int_{0}^{L} xdW = \bar{x}W
\]
\[
(OP)A = \int_{0}^{L} xdA = \bar{x}A
\]
A beam supports a distributed load as shown. Determine the equivalent concentrated load and the reactions at the supports.

**SOLUTION:**

- The magnitude of the concentrated load is equal to the total load or the area under the curve.
- The line of action of the concentrated load passes through the centroid of the area under the curve.
- Determine the support reactions by summing moments about the beam ends.
Sample Problem 5.9

**SOLUTION:**

- The magnitude of the concentrated load is equal to the total load or the area under the curve.

\[ F = 18.0 \text{ kN} \]

- The line of action of the concentrated load passes through the centroid of the area under the curve.

\[ \bar{X} = \frac{63 \text{ kN} \cdot \text{m}}{18 \text{ kN}} \]

\[ \bar{X} = 3.5 \text{ m} \]

### Table

<table>
<thead>
<tr>
<th>Component</th>
<th>( A, \text{ kN} )</th>
<th>( \bar{x}, \text{ m} )</th>
<th>( \bar{x}A, \text{ kN} \cdot \text{m} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triangle I</td>
<td>4.5</td>
<td>2</td>
<td>9</td>
</tr>
<tr>
<td>Triangle II</td>
<td>13.5</td>
<td>4</td>
<td>54</td>
</tr>
<tr>
<td>( \sum A = 18.0 )</td>
<td></td>
<td></td>
<td>( \sum \bar{x}A = 63 )</td>
</tr>
</tbody>
</table>
Sample Problem 5.9

- Determine the support reactions by summing moments about the beam ends.

\[ \sum M_A = 0 : \quad B_y (6 \text{ m}) - (18 \text{ kN})(3.5 \text{ m}) = 0 \]

\[ B_y = 10.5 \text{ kN} \]

\[ \sum M_B = 0 : \quad -A_y (6 \text{ m}) + (18 \text{ kN})(6 \text{ m} - 3.5 \text{ m}) = 0 \]

\[ A_y = 7.5 \text{ kN} \]
Center of Gravity of a 3D Body: Centroid of a Volume

- Center of gravity $G$
  
  $$-W \vec{j} = \sum (-\Delta W \vec{j})$$

  $$\vec{r}_G \times (-W \vec{j}) = \sum [\vec{r} \times (-\Delta W \vec{j})]$$

  $$\vec{r}_G W \times (-\vec{j}) = (\sum \vec{r} \Delta W) \times (-\vec{j})$$

  $$W = \int dW \quad \vec{r}_G W = \int \vec{r} dW$$

- Results are independent of body orientation,
  
  $$\bar{x}W = \int x dW \quad \bar{y}W = \int y dW \quad \bar{z}W = \int z dW$$

- For homogeneous bodies,
  
  $$W = \gamma V \quad \text{and} \quad dW = \gamma dV$$

  $$\bar{x}V = \int x dV \quad \bar{y}V = \int y dV \quad \bar{z}V = \int z dV$$
### Centroids of Common 3D Shapes

<table>
<thead>
<tr>
<th>Shape</th>
<th>( \bar{x} )</th>
<th>Volume</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hemisphere</td>
<td>( \frac{3a}{8} )</td>
<td>( \frac{2}{3} \pi a^3 )</td>
</tr>
<tr>
<td>Semiellipsoid of revolution</td>
<td>( \frac{3h}{8} )</td>
<td>( \frac{2}{3} \pi a^2 h )</td>
</tr>
<tr>
<td>Paraboloid of revolution</td>
<td>( \frac{h}{3} )</td>
<td>( \frac{1}{2} \pi a^2 h )</td>
</tr>
</tbody>
</table>

- **Cone**
  - \( \bar{x} \)
  - \( \frac{h}{4} \)
  - \( \frac{1}{3} \pi a^2 h \)

- **Pyramid**
  - \( \bar{x} \)
  - \( \frac{h}{4} \)
  - \( \frac{1}{3} abh \)
Composite 3D Bodies

- Moment of the total weight concentrated at the center of gravity G is equal to the sum of the moments of the weights of the component parts.

\[
\bar{X} \sum W = \sum \bar{x} W \quad \bar{Y} \sum W = \sum \bar{y} W \quad \bar{Z} \sum W = \sum \bar{z} W
\]

- For homogeneous bodies,

\[
\bar{X} \sum V = \sum \bar{x} V \quad \bar{Y} \sum V = \sum \bar{y} V \quad \bar{Z} \sum V = \sum \bar{z} V
\]
Sample Problem 5.12

SOLUTION:

- Form the machine element from a rectangular parallelepiped and a quarter cylinder and then subtracting two 1-in. diameter cylinders.

Locate the center of gravity of the steel machine element. The diameter of each hole is 1 in.
Sample Problem 5.12

\[ V, \text{ in}^3 \]

<table>
<thead>
<tr>
<th></th>
<th>( \bar{x} ), in</th>
<th>( \bar{y} ), in</th>
<th>( \bar{z} ), in</th>
<th>( \bar{x}V ), in(^4 )</th>
<th>( \bar{y}V ), in(^4 )</th>
<th>( \bar{z}V ), in(^4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>(4.5)(2)(0.5) = 4.5</td>
<td>0.25</td>
<td>-1</td>
<td>2.25</td>
<td>1.125</td>
<td>-4.5</td>
</tr>
<tr>
<td>II</td>
<td>( \frac{1}{4}\pi(2)^2(0.5) = 1.571 )</td>
<td>1.3488</td>
<td>-0.8488</td>
<td>0.25</td>
<td>2.119</td>
<td>-1.333</td>
</tr>
<tr>
<td>III</td>
<td>-( \pi(0.5)^2(0.5) = -0.3927 )</td>
<td>0.25</td>
<td>-1</td>
<td>3.5</td>
<td>-0.098</td>
<td>0.393</td>
</tr>
<tr>
<td>IV</td>
<td>-( \pi(0.5)^2(0.5) = -0.3927 )</td>
<td>0.25</td>
<td>-1</td>
<td>1.5</td>
<td>-0.098</td>
<td>0.393</td>
</tr>
<tr>
<td>( \Sigma V = 5.286 )</td>
<td></td>
<td></td>
<td></td>
<td>( \Sigma \bar{x}V = 3.048 )</td>
<td>( \Sigma \bar{y}V = -5.047 )</td>
<td>( \Sigma \bar{z}V = 8.555 )</td>
</tr>
</tbody>
</table>
Sample Problem 5.12

<table>
<thead>
<tr>
<th>V, in³</th>
<th>x, in.</th>
<th>y, in.</th>
<th>z, in.</th>
<th>xV, in⁴</th>
<th>yV, in⁴</th>
<th>zV, in⁴</th>
</tr>
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<tr>
<td>I</td>
<td>(4.5)(2)(0.5) = 4.5</td>
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</tr>
<tr>
<td>ΣV = 5.286</td>
<td></td>
<td></td>
<td></td>
<td>ΣxV = 3.048</td>
<td>ΣyV = -5.047</td>
<td>ΣzV = 8.555</td>
</tr>
</tbody>
</table>

\[
\bar{X} = \frac{\sum xV}{\sum V} = \left(\frac{3.08 \text{ in}^4}{5.286 \text{ in}^3}\right)
\]

\[
\bar{X} = 0.577 \text{ in.}
\]

\[
\bar{Y} = \frac{\sum yV}{\sum V} = \left(\frac{-5.047 \text{ in}^4}{5.286 \text{ in}^3}\right)
\]

\[
\bar{Y} = 0.577 \text{ in.}
\]

\[
\bar{Z} = \frac{\sum zV}{\sum V} = \left(\frac{1.618 \text{ in}^4}{5.286 \text{ in}^3}\right)
\]

\[
\bar{Z} = 0.577 \text{ in.}
\]