Chapter 8

Principle Stresses Under a Given Loading

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Principle Stresses Under a Given Loading

Introduction

Principle Stresses in a Beam

Sample Problem 8.1

Sample Problem 8.2

Design of a Transmission Shaft

Sample Problem 8.3

Stresses Under Combined Loadings

Sample Problem 8.5
Introduction

• In Chaps. 1 and 2, you learned how to determine the normal stress due to centric loads
  In Chap. 3, you analyzed the distribution of shearing stresses in a circular member due to a twisting couple
  In Chap. 4, you determined the normal stresses caused by bending couples
  In Chaps. 5 and 6, you evaluated the shearing stresses due to transverse loads
  In Chap. 7, you learned how the components of stress are transformed by a rotation of the coordinate axes and how to determine the principal planes, principal stresses, and maximum shearing stress at a point.

• In Chapter 8, you will learn how to determine the stress in a structural member or machine element due to a combination of loads and how to find the corresponding principal stresses and maximum shearing stress
Principle Stresses in a Beam

- Prismatic beam subjected to transverse loading

\[ \sigma_x = -\frac{My}{I} \quad \sigma_m = \frac{Mc}{I} \]
\[ \tau_{xy} = -\frac{VQ}{It} \quad \tau_m = \frac{VQ}{It} \]

- Principal stresses determined from methods of Chapter 7

- Can the maximum normal stress within the cross-section be larger than

\[ \sigma_m = \frac{Mc}{I} \]
**Principle Stresses in a Beam**

<table>
<thead>
<tr>
<th>$y/c$</th>
<th>$x = 2c$</th>
<th>$x = 8c$</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>$\sigma_{\text{min}}/\sigma_m$</td>
<td>$\sigma_{\text{max}}/\sigma_m$</td>
</tr>
<tr>
<td>1.0</td>
<td>0</td>
<td>1.000</td>
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<tr>
<td>0.8</td>
<td>-0.010</td>
<td>0.810</td>
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<tr>
<td>0.6</td>
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<td>0.640</td>
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<td>0.4</td>
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<tr>
<td>0.2</td>
<td>-0.160</td>
<td>0.360</td>
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<tr>
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<tr>
<td>-0.2</td>
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</tr>
</tbody>
</table>

Note: The diagram shows tensile and compressive stresses with arrows indicating the direction and magnitude of the stresses.
• Cross-section shape results in large values of $\tau_{xy}$ near the surface where $\sigma_x$ is also large.

• $\sigma_{max}$ may be greater than $\sigma_m$
Sample Problem 8.1

A 160-kN force is applied at the end of a W200x52 rolled-steel beam.

Neglecting the effects of fillets and of stress concentrations, determine whether the normal stresses satisfy a design specification that they be equal to or less than 150 MPa at section $A-A'$.

SOLUTION:

- Determine shear and bending moment in Section $A-A'$
- Calculate the normal stress at top surface and at flange-web junction.
- Evaluate the shear stress at flange-web junction.
- Calculate the principal stress at flange-web junction
Sample Problem 8.1

SOLUTION:

- Determine shear and bending moment in Section $A-A'$

\[
M_A = (160 \text{kN})(0.375 \text{m}) = 60 \text{kN} \cdot \text{m}
\]

\[
V_A = 160 \text{kN}
\]

- Calculate the normal stress at top surface and at flange-web junction.

\[
\sigma_a = \frac{M_A}{S} = \frac{60 \text{kN} \cdot \text{m}}{512 \times 10^{-6} \text{m}^3} = 117.2 \text{MPa}
\]

\[
\sigma_b = \sigma_a \frac{y_b}{c} = (117.2 \text{MPa}) \frac{90.4 \text{mm}}{103 \text{mm}} = 102.9 \text{MPa}
\]
Sample Problem 8.1

- Evaluate shear stress at flange-web junction.

\[ Q = (204 \times 12.6)96.7 = 248.6 \times 10^3 \text{ mm}^3 \]
\[ = 248.6 \times 10^{-6} \text{ m}^3 \]
\[ \tau_b = \frac{V_A Q}{It} = \frac{(160 \text{ kN})(248.6 \times 10^{-6} \text{ m}^3)}{(52.7 \times 10^{-6} \text{ m}^4)(0.0079 \text{ m})} \]
\[ = 95.5 \text{ MPa} \]

- Calculate the principal stress at flange-web junction

\[ \sigma_{\text{max}} = \frac{1}{2} \sigma_b + \sqrt{\left(\frac{1}{2} \sigma_b\right)^2 + \tau_b^2} \]
\[ = \frac{102.9}{2} + \sqrt{\left(\frac{102.9}{2}\right)^2 + (95.5)^2} \]
\[ = 169.9 \text{ MPa} \quad (> 150 \text{ MPa}) \]

Design specification is not satisfied.
Sample Problem 8.2

The overhanging beam supports a uniformly distributed load and a concentrated load. Knowing that for the grade of steel to be used $\sigma_{all} = 24$ ksi and $\tau_{all} = 14.5$ ksi, select the wide-flange beam which should be used.

**SOLUTION:**

- Determine reactions at $A$ and $D$.
- Determine maximum shear and bending moment from shear and bending moment diagrams.
- Calculate required section modulus and select appropriate beam section.
- Find maximum normal stress.
- Find maximum shearing stress.
Sample Problem 8.2

SOLUTION:

- Determine reactions at $A$ and $D$.
  
  \[ \sum M_A = 0 \quad \Rightarrow \quad R_D = 59 \text{kips} \]
  
  \[ \sum M_D = 0 \quad \Rightarrow \quad R_A = 41 \text{kips} \]

- Determine maximum shear and bending moment from shear and bending moment diagrams.

  \[ |M|_{\text{max}} = 239.4 \text{kip} \cdot \text{in} \quad \text{with} \quad V = 12.2 \text{kips} \]
  
  \[ |V|_{\text{max}} = 43 \text{kips} \]

- Calculate required section modulus and select appropriate beam section.

  \[ S_{\text{min}} = \frac{|M|_{\text{max}}}{\sigma_{\text{all}}} = \frac{24 \text{kip} \cdot \text{in}}{24 \text{ksi}} = 119.7 \text{in}^3 \]

  select W21×62 beam section
Sample Problem 8.2

- Find maximum shearing stress.
  Assuming uniform shearing stress in web,
  \[
  \tau_{\text{max}} = \frac{V_{\text{max}}}{A_{\text{web}}} = \frac{43 \text{ kips}}{8.40 \text{ in}^2} = 5.12 \text{ ksi} < 14.5 \text{ ksi}
  \]

- Find maximum normal stress.
  \[
  \sigma_a = \frac{M_{\text{max}}}{S} = \frac{2873 \text{ kip} \cdot \text{in}}{127 \text{ in}^3} = 22.6 \text{ ksi}
  \]
  \[
  \sigma_b = \sigma_a \frac{y_b}{c} = (22.6 \text{ ksi}) \frac{9.88}{10.5} = 21.3 \text{ ksi}
  \]
  \[
  \tau_b = \frac{V}{A_{\text{web}}} = \frac{12.2 \text{ kips}}{8.40 \text{ in}^2} = 1.45 \text{ ksi}
  \]
  \[
  \sigma_{\text{max}} = \frac{21.3 \text{ ksi}}{2} + \sqrt{\left(\frac{21.3 \text{ ksi}}{2}\right)^2 + (1.45 \text{ ksi})^2} = 21.4 \text{ ksi} < 24 \text{ ksi}
  \]
If power is transferred to and from the shaft by gears or sprocket wheels, the shaft is subjected to transverse loading as well as shear loading.

Normal stresses due to transverse loads may be large and should be included in determination of maximum shearing stress.

Shearing stresses due to transverse loads are usually small and contribution to maximum shear stress may be neglected.
Design of a Transmission Shaft

- At any section,
  \[ \sigma_m = \frac{Mc}{I} \quad \text{where} \quad M^2 = M_y^2 + M_z^2 \]
  \[ \tau_m = \frac{Tc}{J} \]

- Maximum shearing stress,
  \[ \tau_{max} = \sqrt{\left(\frac{\sigma_m}{2}\right)^2 + (\tau_m)^2} = \sqrt{\left(\frac{Mc}{2I}\right)^2 + \left(\frac{Tc}{J}\right)^2} \]
  for a circular or annular cross-section, \( 2I = J \)
  \[ \tau_{max} = \frac{C}{J} \sqrt{M^2 + T^2} \]

- Shaft section requirement,
  \[ \left( \frac{J}{c} \right)_{\text{min}} = \left( \frac{\sqrt{M^2 + T^2}}{\tau_{all}} \right)_{\text{max}} \]
Sample Problem 8.3

SOLUTION:

- Determine the gear torques and corresponding tangential forces.
- Find reactions at $A$ and $B$.
- Identify critical shaft section from torque and bending moment diagrams.
- Calculate minimum allowable shaft diameter.

Solid shaft rotates at 480 rpm and transmits 30 kW from the motor to gears $G$ and $H$; 20 kW is taken off at gear $G$ and 10 kW at gear $H$. Knowing that $\sigma_{all} = 50$ MPa, determine the smallest permissible diameter for the shaft.
Sample Problem 8.3

SOLUTION:

- Determine the gear torques and corresponding tangential forces.

\[
T_E = \frac{P}{2\pi f} = \frac{30\text{kW}}{2\pi(80\text{Hz})} = 597\text{N} \cdot \text{m}
\]

\[
F_E = \frac{T_E}{r_E} = \frac{597\text{N} \cdot \text{m}}{0.16\text{m}} = 3.73\text{kN}
\]

\[
T_C = \frac{20\text{kW}}{2\pi(80\text{Hz})} = 398\text{N} \cdot \text{m} \quad F_C = 6.63\text{kN}
\]

\[
T_D = \frac{10\text{kW}}{2\pi(80\text{Hz})} = 199\text{N} \cdot \text{m} \quad F_D = 2.49\text{kN}
\]

- Find reactions at A and B.

\[
A_y = 0.932\text{kN} \quad A_z = 6.22\text{kN}
\]

\[
B_y = 2.80\text{kN} \quad B_z = 2.90\text{kN}
\]
Sample Problem 8.3

- Identify critical shaft section from torque and bending moment diagrams.

\[
\left( \sqrt{M^2 + T^2} \right)_{\text{max}} = \sqrt{1160^2 + 373^2} + 597^2 \\
= 1357 \text{ N} \cdot \text{m}
\]
Sample Problem 8.3

- Calculate minimum allowable shaft diameter.

\[
\frac{J}{c} = \frac{\sqrt{M^2 + T^2}}{\tau_{all}} = \frac{1357 \text{ N} \cdot \text{m}}{50 \text{ MPa}} = 27.14 \times 10^{-6} \text{ m}^3
\]

For a solid circular shaft,

\[
\frac{J}{c} = \frac{\pi}{2} c^3 = 27.14 \times 10^{-6} \text{ m}^3
\]

\[c = 0.02585 \text{ m} = 25.85 \text{ m}\]

\[d = 2c = 51.7 \text{ mm}\]
Stresses Under Combined Loadings

- Wish to determine stresses in slender structural members subjected to arbitrary loadings.

- Pass section through points of interest. Determine force-couple system at centroid of section required to maintain equilibrium.

- System of internal forces consist of three force components and three couple vectors.

- Determine stress distribution by applying the superposition principle.
Stresses Under Combined Loadings

- Axial force and in-plane couple vectors contribute to normal stress distribution in the section.

- Shear force components and twisting couple contribute to shearing stress distribution in the section.
• Normal and shearing stresses are used to determine principal stresses, maximum shearing stress and orientation of principal planes.

• Analysis is valid only to extent that conditions of applicability of superposition principle and Saint-Venant’s principle are met.
Sample Problem 8.5

Three forces are applied to a short steel post as shown. Determine the principle stresses, principal planes and maximum shearing stress at point $H$.

SOLUTION:

- Determine internal forces in Section $EFG$.
- Evaluate normal stress at $H$.
- Evaluate shearing stress at $H$.
- Calculate principal stresses and maximum shearing stress. Determine principal planes.
Sample Problem 8.5

**SOLUTION:**

- Determine internal forces in Section $EFG$.

\[
V_x = -30 \text{kN} \quad P = 50 \text{kN} \quad V_z = -75 \text{kN}
\]

\[
M_x = (50 \text{kN})(0.130 \text{m}) - (75 \text{kN})(0.200 \text{m}) = -8.5 \text{kN} \cdot \text{m}
\]

\[
M_y = 0 \quad M_z = (30 \text{kN})(0.100 \text{m}) = 3 \text{kN} \cdot \text{m}
\]

**Note:** Section properties,

\[
A = (0.040 \text{m})(0.140 \text{m}) = 5.6 \times 10^{-3} \text{m}^2
\]

\[
I_x = \frac{1}{12}(0.040 \text{m})(0.140 \text{m})^3 = 9.15 \times 10^{-6} \text{m}^4
\]

\[
I_z = \frac{1}{12}(0.140 \text{m})(0.040 \text{m})^3 = 0.747 \times 10^{-6} \text{m}^4
\]
Sample Problem 8.5

- Evaluate normal stress at $H$.

$$\sigma_y = \frac{P}{A} + \frac{|M_z|a}{I_z} - \frac{|M_x|b}{I_z}$$

$$= \frac{50 \text{kN}}{5.6 \times 10^{-3} \text{m}^2} + \frac{(3 \text{kN} \cdot \text{m})(0.020 \text{m})}{0.747 \times 10^{-6} \text{m}^4} - \frac{(8.5 \text{kN} \cdot \text{m})(0.025 \text{m})}{9.15 \times 10^{-6} \text{m}^4}$$

$$= (8.93 + 80.3 - 23.2) \text{MPa} = 66.0 \text{MPa}$$

- Evaluate shearing stress at $H$.

$$Q = A_1\bar{y}_1 = [(0.040 \text{m})(0.045 \text{m})](0.0475 \text{m})$$

$$= 85.5 \times 10^{-6} \text{m}^3$$

$$\tau_{yz} = \frac{V_zQ}{I_xt} = \frac{(75 \text{kN})(85.5 \times 10^{-6} \text{m}^3)}{9.15 \times 10^{-6} \text{m}^4}(0.040 \text{m})$$

$$= 17.52 \text{MPa}$$
Sample Problem 8.5

- Calculate principal stresses and maximum shearing stress.
- Determine principal planes.

\[ \tau_{\text{max}} = R = \sqrt{33.0^2 + 17.52^2} = 37.4 \text{ MPa} \]
\[ \sigma_{\text{max}} = OC + R = 33.0 + 37.4 = 70.4 \text{ MPa} \]
\[ \sigma_{\text{min}} = OC - R = 33.0 - 37.4 = -7.4 \text{ MPa} \]
\[ \tan 2\theta_p = \frac{CY}{CD} = \frac{17.52}{33.0} \quad 2\theta_p = 27.96^\circ \]
\[ \theta_p = 13.98^\circ \]
\[ \tau_{\text{max}} = 37.4 \text{ MPa} \]
\[ \sigma_{\text{max}} = 70.4 \text{ MPa} \]
\[ \sigma_{\text{min}} = -7.4 \text{ MPa} \]
\[ \theta_p = 13.98^\circ \]