CHAPTER 6

MECHANICS OF MATERIALS

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Shearing Stresses in Beams and Thin-Walled Members

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Introduction

• Transverse loading applied to a beam results in normal and shearing stresses in transverse sections.

• Distribution of normal and shearing stresses satisfies

\[ F_x = \int \sigma_x dA = 0 \quad M_x = \int (y \tau_{xz} - z \tau_{xy}) dA = 0 \]
\[ F_y = \int \tau_{xy} dA = -V \quad M_y = \int z \sigma_x dA = 0 \]
\[ F_z = \int \tau_{xz} dA = 0 \quad M_z = \int (-y \sigma_x) = 0 \]

• When shearing stresses are exerted on the vertical faces of an element, equal stresses must be exerted on the horizontal faces.

• Longitudinal shearing stresses must exist in any member subjected to transverse loading.
Shear on the Horizontal Face of a Beam Element

- Consider prismatic beam
- For equilibrium of beam element
  \[ \sum F_x = 0 = \Delta H + \int_A^x (\sigma_D - \sigma_D) \, dA \]
  \[ \Delta H = \frac{M_D - M_C}{I} \int_A^x y \, dA \]
- Note,
  \[ Q = \int_A^x y \, dA \]
  \[ M_D - M_C = \frac{dM}{dx} \Delta x = V \Delta x \]
- Substituting,
  \[ \Delta H = \frac{VQ}{I} \Delta x \]
  \[ q = \frac{\Delta H}{\Delta x} = \frac{VQ}{I} = \text{shear flow} \]
Shear on the Horizontal Face of a Beam Element

- Shear flow,
  \[ q = \frac{\Delta H}{\Delta x} = \frac{VQ}{I} = \text{shear flow} \]

- where
  \[ Q = \int \frac{y}{A} \, dA \]
  is the first moment of area above \( y_1 \)
  \[ I = \int \frac{y^2}{A + A'} \, dA \]
  is the second moment of full cross section

- Same result found for lower area
  \[ q' = \frac{\Delta H'}{\Delta x} = \frac{VQ'}{I} = -q' \]
  \[ Q + Q' = 0 \]
  is the first moment with respect to neutral axis
  \[ \Delta H' = -\Delta H \]
Example 6.01

A beam is made of three planks, nailed together. Knowing that the spacing between nails is 25 mm and that the vertical shear in the beam is \( V = 500 \text{ N} \), determine the shear force in each nail.

SOLUTION:

- Determine the horizontal force per unit length or shear flow \( q \) on the lower surface of the upper plank.
- Calculate the corresponding shear force in each nail.
Example 6.01

\[ Q = A\bar{y} = (0.020 \text{ m} \times 0.100 \text{ m}) (0.060 \text{ m}) = 120 \times 10^{-6} \text{ m}^3 \]

\[ I = \frac{1}{12} (0.020 \text{ m})(0.100 \text{ m})^3 + 2\left[\frac{1}{12} (0.100 \text{ m})(0.020 \text{ m})^3 + (0.020 \text{ m} \times 0.100 \text{ m})(0.060 \text{ m})^2 \right] = 16.20 \times 10^{-6} \text{ m}^4 \]

**SOLUTION:**

- Determine the horizontal force per unit length or shear flow \( q \) on the lower surface of the upper plank.

\[ q = \frac{VQ}{I} = \frac{(500 \text{ N})(120 \times 10^{-6} \text{ m}^3)}{16.20 \times 10^{-6} \text{ m}^4} = 3704 \frac{\text{N}}{\text{m}} \]

- Calculate the corresponding shear force in each nail for a nail spacing of 25 mm.

\[ F = (0.025 \text{ m})q = (0.025 \text{ m})(3704 \text{ N/m}) \]

\[ F = 92.6 \text{ N} \]
The average shearing stress on the horizontal face of the element is obtained by dividing the shearing force on the element by the area of the face.

\[
\tau_{ave} = \frac{\Delta H}{\Delta A} = \frac{q \Delta x}{\Delta A} = \frac{VQ}{I} \frac{\Delta x}{t \Delta x} = \frac{VQ}{It}
\]

On the upper and lower surfaces of the beam, \(\tau_{yx} = 0\). It follows that \(\tau_{xy} = 0\) on the upper and lower edges of the transverse sections.

If the width of the beam is comparable or large relative to its depth, the shearing stresses at \(D_1\) and \(D_2\) are significantly higher than at \(D\).
Shearing Stresses $\tau_{xy}$ in Common Types of Beams

- For a narrow rectangular beam,

$$
\tau_{xy} = \frac{VQ}{Ib} = \frac{3V}{2A} \left(1 - \frac{y^2}{c^2}\right)
$$

$$
\tau_{\text{max}} = \frac{3V}{2A}
$$

- For American Standard (S-beam) and wide-flange (W-beam) beams

$$
\tau_{\text{ave}} = \frac{VQ}{It}
$$

$$
\tau_{\text{max}} = \frac{V}{A_{\text{web}}}
$$
Further Discussion of the Distribution of Stresses in a Narrow Rectangular Beam

- Consider a narrow rectangular cantilever beam subjected to load $P$ at its free end:
  \[ \tau_{xy} = \frac{3}{2} \frac{P}{A} \left(1 - \frac{y^2}{c^2}\right) \quad \sigma_x = +\frac{P_{xy}}{I} \]

- Shearing stresses are independent of the distance from the point of application of the load.

- Normal strains and normal stresses are unaffected by the shearing stresses.

- From Saint-Venant’s principle, effects of the load application mode are negligible except in immediate vicinity of load application points.

- Stress/strain deviations for distributed loads are negligible for typical beam sections of interest.
A timber beam is to support the three concentrated loads shown. Knowing that for the grade of timber used,

\[ \sigma_{all} = 1800 \text{ psi} \quad \tau_{all} = 120 \text{ psi} \]

determine the minimum required depth \( d \) of the beam.

**SOLUTION:**

- Develop shear and bending moment diagrams. Identify the maximums.
- Determine the beam depth based on allowable normal stress.
- Determine the beam depth based on allowable shear stress.
- Required beam depth is equal to the larger of the two depths found.
Sample Problem 6.2

SOLUTION:

Develop shear and bending moment diagrams. Identify the maximums.

\[ V_{\text{max}} = 3 \text{kips} \]
\[ M_{\text{max}} = 7.5 \text{kip} \cdot \text{ft} = 90 \text{kip} \cdot \text{in} \]
Sample Problem 6.2

- Determine the beam depth based on allowable normal stress.
  \[ \sigma_{all} = \frac{M_{\text{max}}}{S} \]
  \[ 1800 \text{ psi} = \frac{90 \times 10^3 \text{ lb} \cdot \text{in.}}{(0.5833 \text{ in.})d^2} \]
  \[ d = 9.26 \text{ in.} \]

- Determine the beam depth based on allowable shear stress.
  \[ \tau_{all} = \frac{3 V_{\text{max}}}{2 A} \]
  \[ 120 \text{ psi} = \frac{3 \times 3000 \text{ lb}}{2 (3.5 \text{ in.})d} \]
  \[ d = 10.71 \text{ in.} \]

- Required beam depth is equal to the larger of the two.
  \[ d = 10.71 \text{ in.} \]
Longitudinal Shear on a Beam Element of Arbitrary Shape

- We have examined the distribution of the vertical components $\tau_{xy}$ on a transverse section of a beam. We now wish to consider the horizontal components $\tau_{xz}$ of the stresses.

- Consider prismatic beam with an element defined by the curved surface CDD’C’.

\[ \sum F_x = 0 = \Delta H + \int_a \left( \sigma_D - \sigma_C \right) dA \]

- Except for the differences in integration areas, this is the same result obtained before which led to

\[ \Delta H = \frac{VQ}{I} \Delta x \]

\[ q = \frac{\Delta H}{\Delta x} = \frac{VQ}{I} \]
Example 6.04

A square box beam is constructed from four planks as shown. Knowing that the spacing between nails is 1.5 in. and the beam is subjected to a vertical shear of magnitude $V = 600$ lb, determine the shearing force in each nail.

SOLUTION:

- Determine the shear force per unit length along each edge of the upper plank.

- Based on the spacing between nails, determine the shear force in each nail.
Example 6.04

SOLUTION:

• Determine the shear force per unit length along each edge of the upper plank.

\[ q = \frac{VQ}{I} = \frac{(600 \text{ lb})(4.22 \text{ in}^3)}{27.42 \text{ in}^4} = 92.3 \text{ lb in} \]

\[ f = \frac{q}{2} = 46.15 \frac{\text{lb}}{\text{in}} \]

= edge force per unit length

• Based on the spacing between nails, determine the shear force in each nail.

\[ F = f \ell = \left(46.15 \frac{\text{lb}}{\text{in}}\right)(1.75 \text{ in}) \]

\[ F = 80.8 \text{ lb} \]

For the upper plank,

\[ Q = A'y = (0.75 \text{ in})(3 \text{ in})(1.875 \text{ in}) \]

= 4.22 in³

For the overall beam cross-section,

\[ I = \frac{1}{12}(4.5 \text{ in})^3 - \frac{1}{12}(3 \text{ in})^3 \]

= 27.42 in⁴
Shearing Stresses in Thin-Walled Members

- Consider a segment of a wide-flange beam subjected to the vertical shear $V$.
- The longitudinal shear force on the element is
  \[ \Delta H = \frac{VQ}{I} \Delta x \]
- The corresponding shear stress is
  \[ \tau_{zx} = \tau_{xz} \approx \frac{\Delta H}{t \Delta x} = \frac{VQ}{It} \]
- Previously found a similar expression for the shearing stress in the web
  \[ \tau_{xy} = \frac{VQ}{It} \]
- NOTE: $\tau_{xy} \approx 0$ in the flanges
  \[ \tau_{xz} \approx 0 \] in the web
Shearing Stresses in Thin-Walled Members

- The variation of shear flow across the section depends only on the variation of the first moment.
  \[ q = \tau t = \frac{VQ}{I} \]

- For a box beam, \( q \) grows smoothly from zero at A to a maximum at C and C’ and then decreases back to zero at E.

- The sense of \( q \) in the horizontal portions of the section may be deduced from the sense in the vertical portions or the sense of the shear \( V \).
Shearing Stresses in Thin-Walled Members

- For a wide-flange beam, the shear flow increases symmetrically from zero at $A$ and $A'$, reaches a maximum at $C$ and the decreases to zero at $E$ and $E'$.

- The continuity of the variation in $q$ and the merging of $q$ from section branches suggests an analogy to fluid flow.
Plastic Deformations

- Recall: \( M_Y = \frac{I}{c} \sigma_Y = \text{maximum elastic moment} \)
- For \( M = PL < M_Y \), the normal stress does not exceed the yield stress anywhere along the beam.
- For \( PL > M_Y \), yield is initiated at \( B \) and \( B' \). For an elastoplastic material, the half-thickness of the elastic core is found from

\[
P x = \frac{3}{2} M_Y \left(1 - \frac{1}{3} \frac{y_Y^2}{c^2}\right)
\]

- The section becomes fully plastic \((y_Y = 0)\) at the wall when

\[
PL = \frac{3}{2} M_Y = M_p
\]
- Maximum load which the beam can support is

\[
P_{\text{max}} = \frac{M_p}{L}
\]
Plastic Deformations

- Preceding discussion was based on normal stresses only.
- Consider horizontal shear force on an element within the plastic zone,
  \[ \Delta H = -(\sigma_C - \sigma_D) dA = -(\sigma_Y - \sigma_Y) dA = 0 \]

Therefore, the shear stress is zero in the plastic zone.

- Shear load is carried by the elastic core,
  \[ \tau_{xy} = \frac{3}{2} \frac{P}{A'} \left(1 - \frac{y^2}{y_Y^2}\right) \]
  where \( A' = 2by_Y \)
  \[ \tau_{\max} = \frac{3}{2} \frac{P}{A'} \]

- As \( A' \) decreases, \( \tau_{\max} \) increases and may exceed \( \tau_Y \).
Sample Problem 6.3

SOLUTION:

• For the shaded area,
  \[ Q = (4.31\text{ in})(0.770\text{ in})(4.815\text{ in}) \]
  \[ = 15.98\text{ in}^3 \]

• The shear stress at \( a \),
  \[ \tau = \frac{VQ}{It} = \frac{(50\text{ kips})(15.98\text{ in}^3)}{(394\text{ in}^4)(0.770\text{ in})} \]
  \[ \tau = 2.63\text{ ksi} \]

Knowing that the vertical shear is 50 kips in a W10x68 rolled-steel beam, determine the horizontal shearing stress in the top flange at the point \( a \).
Unsymmetric Loading of Thin-Walled Members

- Beam loaded in a vertical plane of symmetry deforms in the symmetry plane without twisting.

$$\sigma_x = -\frac{My}{I} \quad \tau_{ave} = \frac{VQ}{It}$$

- Beam without a vertical plane of symmetry bends and twists under loading.

$$\sigma_x = -\frac{My}{I} \quad \tau_{ave} \neq \frac{VQ}{It}$$
Unsymmetric Loading of Thin-Walled Members

- If the shear load is applied such that the beam does not twist, then the shear stress distribution satisfies

\[ \tau_{\text{ave}} = \frac{VQ}{It} \quad V = \int_{B}^{D} q \, ds \quad F = \int_{A}^{B} q \, ds = -\int_{E}^{D} q \, ds = -F' \]

- \( F \) and \( F' \) indicate a couple \( Fh \) and the need for the application of a torque as well as the shear load.

\[ Fh = Ve \]

- When the force \( P \) is applied at a distance \( e \) to the left of the web centerline, the member bends in a vertical plane without twisting.
Example 6.05

- Determine the location for the shear center of the channel section with \( b = 4 \text{ in.} \), \( h = 6 \text{ in.} \), and \( t = 0.15 \text{ in.} \)

\[
e = \frac{F h}{I}
\]

- where

\[
F = \int_0^b q \, ds = \int_0^b VQ \, ds = \frac{V}{I} \int_0^b st \, h \, ds = \frac{Vthb^2}{4I}
\]

\[
I = I_{\text{web}} + 2I_{\text{flange}} = \frac{1}{12}th^3 + 2\left[ \frac{1}{12}bt^3 + bt\left(\frac{h}{2}\right)^2 \right] \\
\approx \frac{1}{12}th^2(6b + h)
\]

- Combining,

\[
e = \frac{b}{2 + \frac{h}{3b}} = \frac{4\text{in.}}{2 + \frac{6\text{in.}}{3(4\text{in.})}} \quad e = 1.6\text{in.}
\]
Example 6.06

- Determine the shear stress distribution for $V = 2.5$ kips.

$$\tau = \frac{q}{t} = \frac{VQ}{It}$$

- Shearing stresses in the flanges,

$$\tau = \frac{VQ}{It} = \frac{V}{It} \left(\frac{st}{h}\right) = \frac{Vh}{2I}$$

$$\tau_B = \frac{Vhb}{2\left(\frac{1}{12}th^2\right)(6b + h)} = \frac{6Vb}{th(6b + h)} = \frac{6(2.5 \text{ kips})(4 \text{ in})}{(0.15 \text{ in})(6 \text{ in})(6 \times 4 \text{ in} + 6 \text{ in})} = 2.22 \text{ ksi}$$

- Shearing stress in the web,

$$\tau_{\text{max}} = \frac{VQ}{It} = \frac{V\left(\frac{1}{8}ht\right)(4b + h)}{\frac{1}{12}th^2(6b + h)t} = \frac{3V(4b + h)}{2th(6b + h)} = \frac{3(2.5 \text{ kips})(4 \times 4 \text{ in} + 6 \text{ in})}{2(0.15 \text{ in})(6 \text{ in})(6 \times 6 \text{ in} + 6 \text{ in})} = 3.06 \text{ ksi}$$