1 Derivative of Composite and Implicit Functions

1.1 Partial Derivative of Composite Function

Theorem 1. Let \( z = f(x, y) \) and \( x = g(t), y = h(t) \) be defined in an appropriate domain and have continuous FIRST PARTIAL DERIVATIVES. Then the composite function \( z(t) \), defined by

\[
z(t) = f[x(t), y(t)] = F(t)
\]
is differentiable and its derivative given by

\[
\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}
\]

(1)

Proof. Let \( t \) be a fixed value and let \( x, y, z \) be the corresponding values of the functions \( g, h \) and \( f \). Then for given \( \Delta t \) (increment of \( t \)), \( \Delta x \) and \( \Delta y \) are determined as

\[
\Delta x = g(t + \Delta t) - g(t)
\]
\[
\Delta y = h(t + \Delta t) - h(t)
\]

while \( \Delta z \) is then determined as

\[
\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)
\]

By the Fundamental lemma, we have

\[
\Delta z = \frac{\partial z}{\partial x} \Delta x + \frac{\partial z}{\partial y} \Delta y + \alpha \Delta x + \beta \Delta y
\]

Hence

\[
\lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} + \alpha \frac{dx}{dt} + \beta \frac{dy}{dt}
\]

that is 1 as required.

Equation 1 is called the derivative of the composite function of the chain rule.

Example 2. Find \( \frac{dz}{dt} \) if \( z = f(x, y) = x^2 + xy, x = e^t \) and \( y = \sin t \).

Solution 3. By using 1 and

\[
\frac{\partial z}{\partial x} = 2x + y, \quad \frac{\partial z}{\partial y} = x, \quad \frac{dx}{dt} = e^t \quad \text{and} \quad \frac{dy}{dt} = \cos t
\]

\[
\frac{dz}{dt} = (2x + y)e^t + x \cos t
\]

\[
\frac{dz}{dt} = (2e^t + \sin t)e^t + e^t \cos t
\]

Note that, the same result can be obtained by the first substituting \( x = e^t \) and \( y = \sin t \) into \( 0x^2 + xy \) and then differentiating

\[
z = e^{2t} + e^t \sin t
\]
\[
\frac{dz}{dt} = 2e^{2t} + e^t \sin t + e^t \cos t = (2e^t + \sin t)e^t + e^t \cos t
\]
Example 4.  a. Use the chain rule to find the derivative of \( z = f(x, y) = xy \) w.r.t. \( t \) along the path \( x = \cos t, \ y = \sin t \).

b. What is the derivative’s value at \( t = \frac{\pi}{2} \)?

Solution 5.  a.

\[
\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}
\]

\[
\frac{\partial z}{\partial x} = y = \sin t, \quad \frac{\partial z}{\partial y} = x = \cos t, \quad \frac{dx}{dt} = -\sin t, \quad \frac{dy}{dt} = \cos t
\]

\[
\frac{dz}{dt} = y(-\sin t) + x\cos t = -\sin t\sin t + \cos t\cos t = -\sin^2 t + \cos^2 t = \cos 2t
\]

In this case we can check the result with a more direct calculation. As a function of \( t \),

\[
z = xy = \cos t\sin t = \frac{1}{2} \sin 2t \text{ so } \frac{dz}{dt} = \frac{d}{dt}\left(\frac{1}{2} \sin 2t\right) = \cos 2t.
\]

b. In either case,

\[
\left(\frac{dz}{dt}\right)_{t=\frac{\pi}{2}} = \cos\left(2, \frac{\pi}{2}\right) = \cos \pi = -1.
\]

Example 6. Let \( z = f(x, y), x = g(t), y = h(t) \). Evaluate the second order derivative \( \frac{d^2z}{dt^2} \) of the composite function.

Solution 7. Let \( x = g(t) = x(t) \) and \( y = h(t) = y(t) \), then \( z = f(x, y) = f(x(t), y(t)) = F(t) \) is composite function. We should use Equation 1.

\[
\frac{d^2z}{dt^2} = \frac{d}{dt}\left(\frac{dz}{dt}\right) = \frac{d}{dt}\left(\frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}\right)
\]

\[
= \frac{d}{dt}\left(\frac{\partial z}{\partial x}\right) \frac{dx}{dt} + \frac{\partial z}{\partial x} \frac{d^2x}{dt^2} + \frac{d}{dt}\left(\frac{\partial z}{\partial y}\right) \frac{dy}{dt} + \frac{\partial z}{\partial y} \frac{d^2y}{dt^2}
\]

Applying Equation 1 for \( \frac{d}{dt}\left(\frac{\partial z}{\partial x}\right) \) and \( \frac{d}{dt}\left(\frac{\partial z}{\partial y}\right) \) we get

\[
\frac{d}{dt}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} \frac{dx}{dt} + \frac{\partial^2 z}{\partial x \partial y} \frac{dy}{dt}
\]

\[
\frac{d}{dt}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} \frac{dy}{dt} + \frac{\partial^2 z}{\partial y \partial x} \frac{dx}{dt}
\]

\[
\frac{d^2z}{dt^2} = \left(\frac{\partial^2 z}{\partial x^2} \frac{dx}{dt} + \frac{\partial^2 z}{\partial x \partial y} \frac{dy}{dt}\right) + \left(\frac{\partial z}{\partial x}\right) \frac{d^2x}{dt^2} + \left(\frac{\partial z}{\partial y}\right) \frac{d^2y}{dt^2} + \left(\frac{\partial^2 z}{\partial y^2} \frac{dy}{dt} + \frac{\partial^2 z}{\partial y \partial x} \frac{dx}{dt}\right)
\]

Theorem 8. Let \( z = f(x, y) \) and \( x = g(u, v), y = h(u, v) \) be defined in an appropriate domain and have continuous first partial derivatives. Then the composite function \( z(u, v) \) is defined by

\[
z(u, v) = f[g(u, v) + h(u, v)] = F(u, v)
\]

differentiable and its derivative given by

\[
\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u}, \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v}
\]

(2)
Proof. If \( v \) be hold fixed, \( x \) and \( y \) reduce to function of the single variable \( u \) and we can apply Theorem 1 at once, obtaining

\[
\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} \tag{3}
\]

Similarly, if \( u \) is hold fixed, \( x \) and \( y \) reduce to function of the single variable \( v \) and we obtain

\[
\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v} \tag{4}
\]

Here [in Equation 3 and 4] all ordinary derivatives in

\[
\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}
\]

now become partial derivatives. Equation 2 is called the derivative of the composite function of the chain rule.

\[\square\]

Example 9. Given the composite function

\[z = f(x, y) = x^2 + y^2, x = u \cos v, y = u \sin v.\]

- Find \( \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v} \).
- Evaluate total(exact) differential of \( z(u, v) \).

Solution 10.

\[
\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} = 2x \cos v + 2y \sin v = 2u \cos^2 v + 2u \sin^2 v = 2u
\]

\[
\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v} = 2x(-u \sin v) + 2y(u \cos v) = -2u^2 \cos v \sin v + 2u^2 \cos v \sin v = 0
\]

- \( dz = \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv = 2udu + 0dv = 2udu \)

Note that, the same result can be obtained by first substituting \( x = u \cos v, y = u \sin v \) into \( z = x^2 + y^2 \) and then differentiating w.r.t. \( u \) and \( v \).

\[z = u^2 \cos^2 v + u^2 \sin^2 v = u^2\]

\[\frac{\partial z}{\partial u} = 2u, \quad \frac{\partial z}{\partial v} = 0\]

1.2 The Chain Rule for the Functions defined on Surfaces

If we are interested in the temperature \( u = f(x, y, z) \) at the points \((x, y, z)\) on a globe in space, we might prefer to think of \( x, y \) and \( z \) as functions of variables \( t \) and \( s \) that give the points longitudes and latitudes. If \( x = g(s, t), y = h(s, t) \) and \( z = k(s, t) \), we could then express the temperature as a function of \( s \) and \( t \) with the composite function

\[u = f[g(s, t), h(s, t), k(s, t)].\]

\( u \) would have partial derivatives w.r.t. both \( s \) and \( t \) that could be calculated in the following way.

If \( u = f(x, y, z), x = g(s, t), y = h(s, t), z = k(s, t) \) are differentiable, then \( u \) is a differentiable function of \( s \) and \( t \) and its partial derivatives are given by the equations

\[
\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial s}
\]

\[
\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial t}
\]
In general, if \( z = f(x, y, t, \ldots) \) and \( x = g(u, v, w, \ldots), y = h(u, v, w, \ldots), t = p(u, v, w, \ldots), \ldots \) then

\[
\begin{align*}
\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial z}{\partial t} \frac{\partial t}{\partial u} + \ldots \\
\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial z}{\partial t} \frac{\partial t}{\partial v} + \ldots \\
\frac{\partial z}{\partial w} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial w} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial w} + \frac{\partial z}{\partial t} \frac{\partial t}{\partial w} + \ldots
\end{align*}
\]

The generalization of formula 1 to the case of a function \( z = f(x_1, x_2, \ldots, x_n) \) whose \( n \) arguments depend on a single variable \( t, x_i = x_i(t), i = 1, 2, \ldots, n \) is given by

\[
\frac{dz}{dt} = \frac{\partial z}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial z}{\partial x_2} \frac{dx_2}{dt} + \cdots + \frac{\partial z}{\partial x_n} \frac{dx_n}{dt} = \sum_{i=1}^{n} \frac{\partial z}{\partial x_i} \frac{dx_i}{dt} \tag{5}
\]

The generalization of Equation 2 to the case of a function \( z = f(x_1, x_2, \ldots, x_n) \) whose \( n \) arguments depend on \( m \) new independent variables \( t_1, \ldots, t_m, x_i = x_i(t_1, t_2, \ldots, t_m), i = 1, 2, \ldots, n \) is

\[
\frac{dz}{dt_j} = \frac{\partial z}{\partial x_1} \frac{dx_1}{dt_j} + \frac{\partial z}{\partial x_2} \frac{dx_2}{dt_j} + \cdots + \frac{\partial z}{\partial x_n} \frac{dx_n}{dt_j} = \sum_{i=1}^{n} \frac{\partial z}{\partial x_i} \frac{dx_i}{dt_j} \quad (j = 1, 2, \ldots, m) \tag{6}
\]

These rules are known as chain rules and are basic for computation of composite functions. Equations 1, 2, 5 are coincide statements of the relations between the derivatives involved. In equation 1, \( z = f[g(t), h(t)] \) is the function of \( t \) whose derivative is denoted by \( \frac{dz}{dt} \), while \( \frac{dx}{dt} \) and \( \frac{dy}{dt} \) stand for \( g'(t) \) and \( h'(t) \), respectively. The derivatives \( \frac{\partial z}{\partial x} \) and \( \frac{\partial z}{\partial y} \), which could be written \( \frac{\partial z}{\partial x} \bigg|_{y} \) and \( \frac{\partial z}{\partial y} \bigg|_{x} \) stand for \( f_x(x, y) \) and \( f_y(x, y) \). In equation 2, \( z = f[g(u, v), h(u, v)] \) is the function whose derivative w.r.t. \( u \) is denoted by \( \frac{\partial z}{\partial u} \), or \( \frac{\partial z}{\partial u} \bigg|_{v} \). A more precise statement of equation 2 would be as follows:

\[
\begin{align*}
\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial u}, \\
\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial v},
\end{align*}
\]

**Example 11.** If \( z = f(x_1, x_2, x_3) = 2x_2^3 - x_2 x_3 + x_1 x_3^2 \) and \( x_1 = 2 \sin t, x_2 = t^2 - t + 1, x_3 = 2e^{-t} \). Find \( \frac{dz}{dt} \) at \( t = 0 \).

**Solution 12.** Since \( x_1 = x_1(t) = 2 \sin t, x_2 = x_2(t) = t^2 - t + 1, x_3 = x_3(t) = 2e^{-t} \) then \( z = f[x_1(t), x_2(t), x_3(t)] = F(t) \) is a composite function of \( t \) and

\[
\begin{align*}
\frac{dz}{dt} &= \frac{\partial z}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial z}{\partial x_2} \frac{dx_2}{dt} + \frac{\partial z}{\partial x_3} \frac{dx_3}{dt} \\
\frac{dz}{dt} &= (4x_1 + x_3^2) \frac{dx_1}{dt} + (-x_3) \frac{dx_2}{dt} + (-x_2 + 3x_1 x_3) \frac{dx_3}{dt}
\end{align*}
\]

The parameter value \( t = 0 \) corresponds to the point \( (x_1 = 0, x_2 = 1, x_3 = 0) \) on the curve \( (x_1(t), x_2(t), x_3(t)) \) at \( t = 0 \)

\[
\begin{align*}
x_1 &= 2 \sin t = 2 \sin 0 = 0 & x_1 &= 0 \\
x_2 &= t^2 - t + 1 = 0^2 - 0 + 1 = 1 & x_2 &= 1 \\
x_3 &= 3e^{-t} = 3e^0 = 3 & x_3 &= 3 \\
(x_1, x_2, x_3) &= (0, 1, 3)
\end{align*}
\]
and

\[
\frac{dx_1}{dt} = 2 \cos t \quad \Rightarrow \quad \frac{dx_1}{dt} \bigg|_{t=0} = 2 \cos t \bigg|_{t=0} = 2 \cos 0 = 2
\]
\[
\frac{dx_2}{dt} = 2t - 1 \quad \Rightarrow \quad \frac{dx_2}{dt} \bigg|_{t=0} = 2t - 1 \bigg|_{t=0} = 2(0) - 1 = -1
\]
\[
\frac{dx_3}{dt} = -3e^{-t} \quad \Rightarrow \quad \frac{dx_3}{dt} \bigg|_{t=0} = -3e^{-t} \bigg|_{t=0} = -3e^{0} = -3
\]

So

\[
\frac{dz}{dt} = (4.0 + 3^2)(2) + (-2)(-1) + (-1 + 3.0.3)(-3) = 18 + 3 + 3 = 24
\]