Effects of the bond dilution on the phase diagrams of a spin-1 transverse Ising model with crystal field interaction on a honeycomb lattice

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A b s t r a c t

A bond diluted spin-1 Ising model with crystal and transverse field interactions is examined for honeycomb lattice by introducing an effective field approximation that takes into account the correlations between different spins that emerge when expanding the identities. The effects of the crystal field as well as the transverse field and dilution on the phase diagrams and order parameter $m_z$ are discussed in detail. A number of interesting and unusual phenomena such as reentrant phenomena and three successive phase transitions originating from the crystal field as well as the transverse field and bond dilution have been found.

1. Introduction

Ising model in a transverse field (TIM) has been widely examined in statistical mechanics and condensed matter physics since the pioneering work of de Gennes [1] who introduced it as a pseudo-spin model for hydrogen-bonded ferroelectrics such as the KH$_2$PO$_4$ type. Following studies have been predicted that this semi-quantum mechanical model can be successfully applied to a variety of physical systems such as DyVO$_4$, TbVO$_4$ [2] and some real magnetic materials [3]. Hence, the early experimental results corroborate many theoretical predictions. Thereby, many authors have paid considerable attention to this simple model and the problem has been investigated by various theoretical methods [4–10]. Although these experimental observations are consistent with the theoretical studies mentioned above, subsequent experiments made on various real magnetic materials pointed out that the usual model requires some extensions including the crystal field interactions and disorder effects such as site and bond dilution. However, there are a few studies in the literature that include the crystal field as well as the transverse field interactions. Recently, the effect of both the transverse field and the crystal field on the spin-$S$ Ising model with spins of magnitude $S \geq 1$ have been studied and it has been shown that TIM model presents a rich variety of critical phenomena [11–17].

Furthermore, after the observation of reentrant magnetism by Maletta and Felsh [18], the interest of the authors has been directed to investigation of the disorder effects on the thermal and magnetic properties of TIM where the type of the bond interactions is chosen at random according to a given probability distribution. From the theoretical point of view, these systems have been studied by a variety of techniques such as renormalization group method (RG) [19], effective field theory (EFT) [20–34], finite cluster approximation (FCA) [35–39], finite cluster renormalization group (FCR) [40], mean field renormalization group (MFRG) [41], Monte Carlo simulations (MC) [42,43], series expansion (SE) [44] and two spin cluster approximation (TSCA) [45–47], as well as some basic studies based on exact calculations [48,49]. EFT has been widely used in the literature and some interesting results have been reported. For example, Jiang [24] has studied a bond diluted spin-1
system on a honeycomb lattice with crystal field and transverse field interactions and found that the reentrant phenomenon of second order occurs for the system with specific values of the Hamiltonian parameters. The condition for the occurrence of the tricritical point was also given. Jiang et al. [31] discussed the phase diagrams of a random nearest-neighbor interaction spin-1 Ising model with a crystal field in an applied transverse field on the honeycomb lattice. They found two different types of reentrant phenomena due to the competition between the random bond and the negative crystal field parameter. Recently, Yan and Deng [32] reported some unusual phenomena such as three successive phase transitions with a certain range of crystal field and concentration of bond dilution for the bond diluted spin-1 transverse ferromagnetic Ising model with a random crystal field. They argued that this unusual phenomena may be attributed to the randomness of the crystal field. More recently, Bouziane and Saber [21] investigated the phase diagrams and order parameters of the transverse spin-1 Ising model with random interactions for a simple cubic lattice.

Conventional EFT approximations include spin–spin correlations resulting from the usage of the Van der Waerden identities [33] where \( \langle \cdots \rangle \) denotes the thermal average of the central spin is equal to that of its nearest-neighbor spins. The Hamiltonian describing our model is

\[
H = - \sum_{\langle i,j \rangle} J_{ij} S_i^z S_j^z - D \sum_i (S_i^z)^2 - \Omega \sum_i S_i^z,
\]

(1)

where \( S_i^z \) and \( S_i^x \) denote the \( z \) and \( x \) components of the spin operator, respectively. The first summation in Eq. (1) is over the nearest-neighbor pairs of spins and the operator \( S_i^z \) takes the values \( S_i^z = 0, \pm 1 \). \( J_{ij}, D \) and \( \Omega \) terms stand for the exchange interaction, single-ion anisotropy (i.e. crystal field) and transverse field, respectively. We assume that the nearest-neighbor interactions are randomly distributed on the lattice according to the probability distribution function

\[
P(J_{ij}) = (1-p)\delta(J_{ij}) + p\delta(J_{ij} - J),
\]

(2)

where \( 0 < p \leq 1 \) and it denotes the concentration of closed bonds. We construct the mathematical background of our model by using the approximated spin correlation identities [34] by taking into account random configurational averages

\[
\langle (f_i S_i^z)^\alpha \rangle_r = \frac{\text{Tr}_r S_i^z \exp(-\beta H)}{\text{Tr}_r \exp(-\beta H)},
\]

\[
\langle (f_i S_i^z)^2 \rangle_r = \frac{\text{Tr}_r (S_i^z)^2 \exp(-\beta H)}{\text{Tr}_r \exp(-\beta H)},
\]

(3)

(4)

where \( \beta = 1/k_B T \), \( \alpha = z \) or \( x \), \( \{ f_i \} \) is an arbitrary function which is independent of the central spin \( S_i \) and the inner \( \langle \cdots \rangle \) and the outer \( \langle \cdot \cdot \cdot \rangle \), product represent the thermal and random configurational averages, respectively.

In order to apply the differential operator technique [35,36], we should separate the Hamiltonian (1) into two parts as \( H = H_i + H' \). Here, the effective Hamiltonian \( H_i \) includes all the contributions associated with site \( i \), and the other part \( H' \) does not depend on site \( i \). At this point, one should notice that \( H_i \) and \( H' \) do not commute with each other. We can write \(-H_i\) as

\[
-H_i = E_i S_i^z + D (S_i^z)^2 + \Omega S_i^z,
\]

(5)

where \( E_i = \sum_{j \neq i} J_{ij} S_j^z \) is the local field on the site \( i \). If we use the matrix representations of the operators \( S_i^z \) and \( S_i^z \) for the spin-1 system then we can obtain the matrix form of Eq. (5)

\[
-H_i = \begin{pmatrix}
E_i + D & \Omega/\sqrt{2} & 0 \\
\Omega/\sqrt{2} & 0 & \Omega/\sqrt{2} \\
0 & \Omega/\sqrt{2} & -E_i + D
\end{pmatrix}.
\]

(6)
In order to proceed further, we have to diagonalize \(-H_1\) matrix in Eq. (6). The three eigenvalues are

\[
\begin{align*}
\lambda_1 &= \frac{2D}{3} + \frac{2\rho}{3} \cos \left( \frac{\theta}{3} \right), \\
\lambda_2 &= \frac{2D}{3} - \frac{2\rho}{3} \cos \left( \frac{\pi - \theta}{3} \right), \\
\lambda_3 &= \frac{2D}{3} - \frac{2\rho}{3} \cos \left( \frac{\pi + \theta}{3} \right),
\end{align*}
\]

(7)

where

\[
\theta = \arccos \left( \frac{\xi}{\rho^3} \right),
\]

\[
\xi = D \left( 9E_i^2 - \frac{9}{2} \Omega^2 - D^2 \right),
\]

\[
\rho^2 = 3E_i^2 + 3\Omega^2 + D^2,
\]

and the eigenvectors \(\phi_k\) of \(-H_1\) corresponding to the eigenvalues in Eq. (7) are calculated as follows

\[
\begin{align*}
\alpha_k &= \pm \sqrt{1 - \beta_k^2 - \gamma_k^2}, \\
\beta_k &= - \frac{[E_i + (D - \lambda_k)]}{\Omega / \sqrt{2}} \alpha_k, \\
\gamma_k &= - \frac{[E_i + (D - \lambda_k)]}{[E_i - (D - \lambda_k)]} \alpha_k, \\
\phi_k &= \begin{pmatrix} \alpha_k \\ \beta_k \\ \gamma_k \end{pmatrix}, \quad k = 1, 2, 3.
\end{align*}
\]

(8)

Hereafter, we apply the differential operator technique in Eqs. (3) and (4) with \(\{f_i\} = 1\). From Eq. (3) we obtain the following spin correlations for the thermal and configurational averages of a central spin for a honeycomb lattice \((q = 3)\) as

\[
\begin{align*}
\langle S_0^z \rangle_r &= \left\langle \prod_{j=1}^{q-1} \left[ 1 + S_j^z \sinh (J_{ij} \nabla) + (S_j^z)^2 \cosh (J_{ij} \nabla) - 1 \right] \right\rangle F(x)_{|x=0}, \\
\langle S_0^z \rangle_r &= \left\langle \prod_{j=1}^{q-1} \left[ 1 + S_j^z \sinh (J_{ij} \nabla) + (S_j^z)^2 \cosh (J_{ij} \nabla) - 1 \right] \right\rangle H(x)_{|x=0}.
\end{align*}
\]

(9)  (10)

With the help of Eqs. (7) and (8) the functions \(F(x)\) and \(H(x)\) can be calculated from

\[
F(x) = \frac{1}{3} \sum_{n=1}^{3} \exp(\beta_{\lambda_n}) \sum_{n=1}^{3} \langle \phi_n | S_0^z | \phi_n \rangle \exp(\beta_{\lambda_n}),
\]

(11)

\[
H(x) = \frac{1}{3} \sum_{n=1}^{3} \exp(\beta_{\lambda_n}) \sum_{n=1}^{3} \langle \phi_n | S_0^z | \phi_n \rangle \exp(\beta_{\lambda_n}).
\]

(12)

By expanding the right-hand sides of Eqs. (9) and (10) we get the longitudinal and transverse spin correlations as

\[
\begin{align*}
m_z &= \langle S_0^z \rangle_r = l_0 + 3k_1 \langle S_1^z \rangle_r + 3(l_1 - l_0) \langle S_1^z \rangle_r + 3l_2 \langle S_1 S_2 \rangle_r + 6(k_2 - k_1) \langle S_1 S_2^2 \rangle_r + 3(l_0 - 2l_1 + l_2) \langle S_1^2 S_2 \rangle_r + 3l_2 \langle S_1 S_2 S_3 \rangle_r + 3(l_4 - l_2) \langle S_1 S_2 S_3 \rangle_r + 3(k_1 - 2k_2 + k_4) \langle S_1 S_2^2 S_3 \rangle_r + (-l_0 + 3l_1 - l_3 + l_5) \langle S_1 S_2^2 S_3 \rangle_r,
\end{align*}
\]

(13)

\[
\begin{align*}
m_x &= \langle S_0^x \rangle_r = p_0 + 3c_1 \langle S_1 \rangle_r + 3(p_1 - p_0) \langle S_1^2 \rangle_r + 3p_2 \langle S_1 S_2 \rangle_r + 6(c_2 - c_1) \langle S_1 S_2^2 \rangle_r + 3(p_0 - 2p_1 + p_3) \langle S_1^2 S_2 \rangle_r + 3c_1 \langle S_1 S_2 S_3 \rangle_r + 3(p_4 - p_3) \langle S_1 S_2 S_3 \rangle_r + 3(c_1 - 2c_2 + c_4) \langle S_1 S_2^2 S_3 \rangle_r + (-p_0 + 3p_1 - 3p_3 + p_5) \langle S_1 S_2^2 S_3 \rangle_r.
\end{align*}
\]

(14)

We note that, for the sake of simplicity, the superscript \(z\) is omitted from the correlation functions on the right-hand side of Eqs. (13) and (14). The coefficients in Eqs. (13) and (14) are defined as follows
with the coefficients
\begin{align}
a_1 &= F(\gamma), \\
a_2 &= \langle \sinh(J_{ij}\nabla)\rangle_r F(x + \gamma)|_{x=0}, \\
a_3 &= \langle \cosh(J_{ij}\nabla)\rangle_r F(x + \gamma)|_{x=0},
\end{align}
where \( \gamma = (q - 1)\alpha \) is the effective field produced by the \( (q - 1) \) spins outside the system and \( A \) is an unknown parameter to be determined self-consistently. In the effective field approximation, the number of independent spin variables describes the considered system. This number is given by the relation \( \nu = \langle \langle S^z_i \rangle^2 \rangle_r \). As an example for the spin-1 system, \( 2S = 2 \) which means that we have to introduce the additional parameters \( \langle \langle S^z_i \rangle^2 \rangle_r \), resulting from the usage of the Van der Waerden identity for the spin-1 Ising system. With the help of Eq. (4)

\begin{align}
\langle \langle S^z_i \rangle^2 \rangle_r &= \left( \prod_{j=1}^{d} \left[ 1 + S_j^z \sinh(J_{j}\nabla) + (S_j^z)^2 \{\cosh(J_{j}\nabla) - 1\} \right] \right) G(x)|_{x=0}, \\
\langle \langle S^z_i \rangle^2 \rangle_r &= \left( \prod_{j=1}^{d} \left[ 1 + S_j^z \sinh(J_{j}\nabla) + (S_j^z)^2 \{\cosh(J_{j}\nabla) - 1\} \right] \right) K(x)|_{x=0},
\end{align}

where the functions \( G(x) \) and \( K(x) \) are defined as

\begin{align}
G(x) &= \frac{1}{3} \sum_{n=1}^{3} \left\langle \psi_n | (S^z_i)^2 | \psi_n \right\rangle \exp(\beta n), \\
K(x) &= \frac{1}{3} \sum_{n=1}^{3} \left\langle \psi_n | (S^z_i)^2 | \psi_n \right\rangle \exp(\beta n).
\end{align}

Hence, we get the quadrupolar moments by expanding the right-hand sides of Eqs. (20) and (21)

\begin{align}
\langle \langle S^z_i \rangle^2 \rangle_r &= r_0 + 3n_1 \langle \langle S^z_i \rangle^2 \rangle_r + 3(r_1 - n_0) \langle \langle S^z_i \rangle^2 \rangle_r + 3r_2 \langle \langle S^z_i \rangle^2 \rangle_r + 6(n_2 - n_1) \langle \langle S^z_i \rangle^2 \rangle_r + 3(n_1 - n_2 + n_2) \langle \langle S^z_i \rangle^2 \rangle_r + 3(n_1 - n_2 + n_3) \langle \langle S^z_i \rangle^2 \rangle_r + 3(n_1 - 2n_1 + n_2) \langle \langle S^z_i \rangle^2 \rangle_r + \langle \langle S^z_i \rangle^2 \rangle_r \\
\langle \langle S^z_i \rangle^2 \rangle_r &= v_0 + 3n_1 \langle \langle S^z_i \rangle^2 \rangle_r + 3(v_1 - v_0) \langle \langle S^z_i \rangle^2 \rangle_r + 3v_2 \langle \langle S^z_i \rangle^2 \rangle_r + 3(n_2 - n_1) \langle \langle S^z_i \rangle^2 \rangle_r + 3(v_0 - 2v_1 + v_3) \langle \langle S^z_i \rangle^2 \rangle_r + 3(n_1 - 2n_1 + n_2) \langle \langle S^z_i \rangle^2 \rangle_r + 3(n_1 - 2n_1 + n_3) \langle \langle S^z_i \rangle^2 \rangle_r + 3(n_1 - 2n_1 + n_2) \langle \langle S^z_i \rangle^2 \rangle_r + \langle \langle S^z_i \rangle^2 \rangle_r,
\end{align}

with
\[ r_0 = G(0), \]
\[ r_1 = (\cosh(J_0 \nabla))_r G(x)|_{x=0}, \quad n_1 = (\sinh(J_0 \nabla))_r G(x)|_{x=0}, \]
\[ r_2 = (\sinh(J_0 \nabla))^2_r G(x)|_{x=0}, \quad n_2 = (\cosh(J_0 \nabla))(\sinh(J_0 \nabla))_r G(x)|_{x=0}, \]
\[ r_3 = (\cosh(J_0 \nabla))^2_r G(x)|_{x=0}, \quad n_3 = (\sinh(J_0 \nabla))^2_r G(x)|_{x=0}, \]
\[ r_4 = (\cosh(J_0 \nabla))_r (\sinh(J_0 \nabla))^2_r G(x)|_{x=0}, \quad n_4 = (\cosh(J_0 \nabla))^2_r (\sinh(J_0 \nabla))_r G(x)|_{x=0}. \]
\[ r_5 = (\cosh(J_0 \nabla))^3_r G(x)|_{x=0}, \]
\[ v_0 = K(0), \]
\[ v_1 = (\cosh(J_0 \nabla))_r K(x)|_{x=0}, \quad \mu_1 = (\sinh(J_0 \nabla))_r K(x)|_{x=0}, \]
\[ v_2 = (\sinh(J_0 \nabla))^2_r K(x)|_{x=0}, \quad \mu_2 = (\cosh(J_0 \nabla))(\sinh(J_0 \nabla))_r K(x)|_{x=0}, \]
\[ v_3 = (\cosh(J_0 \nabla))^2_r K(x)|_{x=0}, \quad \mu_3 = (\sinh(J_0 \nabla))^2_r K(x)|_{x=0}, \]
\[ v_4 = (\cosh(J_0 \nabla))_r (\sinh(J_0 \nabla))^2_r K(x)|_{x=0}, \quad \mu_4 = (\cosh(J_0 \nabla))^2_r (\sinh(J_0 \nabla))_r K(x)|_{x=0}, \]
\[ v_5 = (\cosh(J_0 \nabla))^3_r K(x)|_{x=0}. \hspace{1cm} (26) \]

Corresponding to Eq. (17)
\[ \langle (S^z_0)^2 \rangle_r = \langle 1 + S^z_0 \sinh(J_0 \nabla) + (S^z_0)^2 \{ \cosh(J_0 \nabla) - 1 \} \rangle_r G(x + \gamma), \]
\[ \langle S^z_i \rangle_r = b_1(1 - \langle (S^z_0)^2 \rangle_r) + b_2 \langle S^z_0 \rangle_r + b_3 \langle (S^z_0)^2 \rangle_r, \hspace{1cm} (29) \]

where
\[ b_1 = G(0), \]
\[ b_2 = (\sinh(J_0 \nabla))_r G(x + \gamma)|_{x=0}, \]
\[ b_3 = (\cosh(J_0 \nabla))_r G(x + \gamma)|_{x=0}. \]
\[ \langle (S^z_i)^2 \rangle_r \equiv \langle (S^z_0)^2 \rangle_r \langle (S^z_0)^2 \rangle_r \cdots \langle (S^z_0)^2 \rangle_r, \hspace{1cm} (32) \]

for \( i \neq j \neq \cdots \neq I [56]. \) The main difference of the method used in this study from the other approximations in the literature emerges in comparison with any decoupling approximation (DA) when expanding the right-hand sides of Eqs. (9), (10), (20) and (21) are expanded, the multi-spin correlation functions appear. The simplest approximation, and one of the most frequently adopted is to decouple these correlations according to

\[ \langle (S^z_i)^2 \cdots S^z_I \rangle_r \equiv \langle S^z_i \rangle_r \langle (S^z_0)^2 \rangle_r \cdots \langle (S^z_0)^2 \rangle_r, \]

Eqs. (13), (14), (18), (24), (25) and (29) are the fundamental correlation functions of the system. When the right-hand sides of Eqs. (9), (10), (20) and (21) are expanded, the multi-spin correlation functions appear. The simplest approximation, and one of the most frequently adopted is to decouple these correlations according to

\[ \langle (S^z_i)^2 \cdots S^z_I \rangle_r \equiv \langle S^z_i \rangle_r \langle (S^z_0)^2 \rangle_r \cdots \langle (S^z_0)^2 \rangle_r, \]

for \( i \neq j \neq \cdots \neq I [56]. \) The main difference of the method used in this study from the other approximations in the literature emerges in comparison with any decoupling approximation (DA) when expanding the right-hand sides of Eqs. (9), (10), (20) and (21). In other words, one advantage of the approximation method used in this study is that no uncontrolled decoupling procedure is used for the higher order correlation functions.

For spin-1 Ising system with \( q = 3, \) taking Eqs. (13), (14), (18), (24), (25) and (29) as a basis, we derive a set of linear equations of the spin correlation functions which interact with the system. At this point, we assume that (i) the correlations depend only on the distance between the spins, (ii) the average values of a central spin and its nearest-neighbor spin (it is labeled as the perimeter spin) are equal to each other with the fact that, in the matrix representations of spin operator \( \hat{S}, \) the spin-1 system has the properties \((S^z_i)^3 = S^z_i \) and \((S^z_i)^4 = (S^z_i)^2. \) Thus, the number of the set of linear equations obtained for the spin-1 Ising system with \( q = 3 \) reduces to twenty three and the complete set is given in Appendix.

If Eq. (A.1) is written in the form of a 23 \( \times \) 23 matrix and solved in terms of the variables \( x_i [(i = 1, 2, \ldots, 23)](e.g., x_1 = \langle S^z_0 \rangle_r, x_2 = \langle S^z_5 \rangle_r, \ldots, x_{23} = \langle (S^z_0)^2 \rangle_r) \) of the linear equations, all of the spin correlation functions can be easily determined as functions of the temperature, effective field, crystal field, transverse magnetic field and bond concentration which the other studies in the literature do not include. Since the thermal and configurational averages of the central spin is equal to that of its nearest-neighbor spins within the present method then the unknown parameter \( A \) can be numerically determined by the relation

\[ \langle S^z_0 \rangle_r = \langle S^z_1 \rangle_r \quad \text{or} \quad x_1 = x_4. \hspace{1cm} (33) \]
Table 1
Bond percolation threshold value $p_c$ for $D/J = 0$ and $\Omega/J = 0$ obtained by several methods and present work for a honeycomb lattice.

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Fig. 1. Variation of the bond percolation threshold $p_c$ with crystal field $D/J$ and transverse field $\Omega/J$. (a) 3D contour plot surface, (b) projection on $(\Omega/J-D/J)$ plane.

By solving Eq. (33) numerically at a given fixed set of Hamiltonian parameters we obtain the parameter $A$. Then we use the numerical values of $A$ to obtain the spin correlation functions $\langle S^z_i \rangle$, $\langle S^x_i \rangle$, (longitudinal and transverse magnetizations), $\langle (S^z_i)^2 \rangle$, $\langle (S^x_i)^2 \rangle$, (longitudinal and transverse quadrupolar moments) and so on, which can be found from Eq. (A.1). Note that $A = 0$ is always the root of Eq. (33) corresponding to the disordered state of the system. The nonzero root of $A$ in Eq. (33) corresponds to the long range ordered state of the system. Once the spin correlation functions have been evaluated then we can give the numerical results for the thermal and magnetic properties of the system. Since the effective field $\gamma$ is very small in the vicinity of $k_B T_c/J$, we can obtain the critical temperature for the fixed set of Hamiltonian parameters by solving Eq. (33) in the limit of $\gamma \to 0$ then we can construct the whole phase diagrams of the system. Another critical parameter is the bond percolation threshold $p_c$ defined as the critical bond concentration value at which the critical temperature reduces to zero. For the selected values of the crystal field $D/J$ and transverse field $\Omega/J$, this critical value can be obtained by solving Eq. (33) in the limits of $\gamma \to 0$ and $k_B T_c/J \to 0$.

3. Results and discussion

In this section, we discuss the phase diagrams of the system one of which is the variation of the critical temperature $k_B T_c/J$ with one Hamiltonian parameter, while the other two parameters are fixed and the other one is the variation of the bond percolation threshold $p_c$ with $D/J$ and $\Omega/J$. Also, in order to clarify the type of the transitions in the system, we give the temperature dependence of the order parameter $m_z$. In Fig. 1, we plot the dependence of the bond percolation threshold surface with $-1.0 < D/J < 5$ and $0.1 < \Omega < 1.2$. As we can see from Fig. 1(a), the effect of the transverse field $\Omega/J$ on the percolation threshold value clearly depends on the value of the crystal field $D/J$ and vice versa. Namely, for the values of $-1.0 < D/J < 0.32$, as $\Omega/J$ increases then the $p_c$ increases and reaches its maximum value. On the other hand, for the other $D/J$ values, $p_c$ decreases for a while and then tends to increase with increasing $\Omega/J$. If the value of $D/J$ is sufficiently positive then $p_c$ value remains more or less constant being independent from the value of $\Omega/J$. We think that the mechanism underlying this behavior completely originates from a collective effect of both $\Omega/J$ and $D/J$ or strictly speaking, we can mention about a competition between $D/J$ and $\Omega/J$ on the system. One should notice that, for the values of $D/J = -1.0$ and $\Omega/J = 1.2$ the system cannot exhibit a long range ordering since $p_c = 1.0$. Moreover, in the absence of crystal and transverse field interactions, numerical value of percolation threshold $p_c$ is given in Table 1 and it is compared for the other works in the literature. As seen from Table 1, we improve the results of the other works based on EFT approximation [30,32,35,37].

In Fig. 2, we plot the phase diagrams in ($k_B T_c/J$–$D/J$) plane for $\Omega/J = 0, 0.5, 1.0$ and 1.5 where the solid and dotted lines correspond to the second and first order transitions and solid circles denote the tricritical points. The numbers accompanying each curve denote the value of bond concentration $p$. At first sight, it is obvious that applying a transverse field $\Omega/J$ and diluting the nearest-neighbor bonds reduce the critical temperature and destruct the first order transitions in the system. However, increasing the $\Omega/J$ value causes the ferromagnetic state to disappear for lower bond concentrations. As seen in the upper left panel in Fig. 2 where $\Omega/J = 0$, the reentrant behavior can be observed for both positive and negative $D/J$ values.
Fig. 2. The phase diagrams of the system in \((k_B T_c/J–D/J)\) plane with selected values of transverse field (a) \(\Omega/J = 0\), (b) \(\Omega/J = 0.5\), (c) \(\Omega/J = 1.0\) and (d) \(\Omega/J = 1.5\). Solid and dotted curves correspond to the second and first order phase transitions, respectively. Solid circles represent the tricritical points. The numbers on the curves denote the bond concentration \(p\).

Fig. 3. Two examples of the phase diagrams in \((k_B T_c/J–D/J)\) plane exhibiting reentrant phase transition. The solid and dotted lines correspond to the second and first order transitions and solid circle represents the tricritical point.

depending on the bond concentration value \(p\). Namely, for \(0.54 < p < 0.77\) the system exhibits a second order reentrant behavior for positive crystal field values, while for \(0.85 < p \leq 1.0\) the system exhibits a tricritical point which decreases as \(p\) decreases. After a specific concentration such as \(p < 0.85\), tricritical point disappears and it turns into second order reentrant transition. Furthermore, applying a transverse field \(\Omega/J\) tends to remove reentrant phase transitions for positive and negative valued crystal fields which can be seen from the upper right panel in Fig. 2. For sufficiently high transverse field values such as \(\Omega/J = 1.0\) and \(1.5\) there is no reentrant behavior in the system (see lower panels in Fig. 2). For \(p = 1.0\), this value is obtained as \(\Omega/J = 0.861\). We note that, there are two different types of reentrant behavior in the system which are depicted in Fig. 3.

Fig. 4 shows the effect of the bond concentration \(p\) on the phase diagrams in \((k_B T_c/J–\Omega/J)\) plane for selected values of \(D/J\). For \(D/J = -1.1\), there is no first order phase transition for \(p < 0.9\). In our calculations, we realized that one can
Fig. 4. The phase diagrams of the system in \((k_B T_c/J - \Omega/J)\) plane with selected values of the crystal field (a) \(D/J = -1.1\), (b) \(D/J = -0.5\), (c) \(D/J = 0\) and (d) \(D/J = 0.5\). Solid and dotted curves correspond to the second and first order phase transitions, respectively. The numbers on the curves denote the bond concentration \(p\).

Fig. 5. The phase diagram in \((k_B T_c/J - \Omega/J)\) plane with \(D/J = 0.48\) and for a range of bond concentration \(0.62 \leq p \leq 0.68\).

Observe the reentrant behavior in the system for the values of \(\Omega/J < 0.37\) with \(D/J = -1.1\). Besides, for the values of \(p \leq 0.79\) the transition lines exhibit a bulge which gets smaller as the value of \(p\) approaches the value of 0.76 which means that ferromagnetic phase region gets narrower. For \(D/J = 0.5\) (the lower right panel in Fig. 4), we can call attention to the signs of an interesting behavior, i.e. three successive phase transition points for the specific values of \(p\). In this system three successive phase transitions can be observed for \(0.6 \leq p \leq 0.7\) in the range \(0.07 \leq \Omega/J \leq 0.32\) and \(0.08 \leq D/J \leq 0.97\). In order to investigate this extraordinary behavior in detail, we examine the phase diagram with \(D/J = 0.48\) and for several values of \(p\) in Fig. 5. We see that \(\Omega/J\) has a very complex effect on the mechanism of reentrant behavior, as stated in Ref. [31]. It is also shown in Fig. 5 that, reentrant behavior disappears for \(p < 0.65\) with \(D/J = 0.48\). In a recent paper, Yan and Deng [22,32] attributed the three successive phase transition behaviors to the randomness of the crystal fields. But, as we can clearly see from Fig. 5 that this phenomena can occur for the systems which can be represented with a Hamiltonian defined in Eq. (1).
In Fig. 6, the effect of the transverse field $\Omega/J$ on the phase diagrams in $(k_B T_c/J-p)$ plane is shown for some selected values of the crystal field $D/J$. As seen in this figure, for $D/J = -1.1$ and $\Omega/J = 0$ the system exhibits a tricritical behavior and its critical temperature cannot reach zero, hence we cannot speak on any percolation threshold value. In addition, we observe a reentrant behavior of second order for $\Omega/J = 0.25$, 0.5, 0.75, although it is hard to comment on this naive picture about reentrant behavior. Therefore, in order to clarify the situation, the phenomenon is shown in Fig. 7 in detail for $\Omega/J = 0.5$ and $D/J = -1.1$. For $D/J = -1.1$ and $-0.5$ in Fig. 6, as the bond concentration $p$ decreases starting from 1.0 then the critical temperature of the system decreases and reaches zero at the bond percolation threshold (except the curve with $D/J = -1.1$ and $\Omega/J = 0$), hence the ferromagnetic region gets narrower. On the other hand, for $D/J \geq 0$ percolation threshold increases and makes the ferromagnetic region wider with increasing transverse field for a while then it tends to decrease and hence the ferromagnetic region becomes narrower this time. This behavior agrees with the observations noted in Fig. 1. We also note that reentrant behavior and three successive transitions can be seen in Fig. 6.
Fig. 8. Temperature dependence of the longitudinal component of magnetization $m_z$ as functions of Hamiltonian parameters $D/J$, $\Omega/J$ and $p$. Solid and dashed magnetization curves exhibit second and first order phase transition properties, respectively.

Finally in Fig. 8, in order to clarify the type of the transitions in the system, we plot the temperature dependence of the order parameter $m_z$. Four different types of transition profiles are shown for several values of Hamiltonian parameters. For example, for Hamiltonian parameter values $p = 0.7$, $\Omega/J = 0$ and $D/J = -0.7$, $-0.9$ as the temperature increases starting from zero then the magnetization curve gradually decreases and reduces to zero at the second order transition temperature. As an example of second order reentrant behavior, we can take a look at the magnetization curves that exist in a second order reentrant regime. We see that the longitudinal magnetization $m_z$ of the system exhibits two critical temperatures of the second order for $p = 0.8$, $D/J = -1.1$ and $\Omega/J = 0.15$. For the other type of reentrant behavior in which a first order transition is followed by a second order transition, the magnetization curves can be seen in the lower right panel of Fig. 8 with Hamiltonian parameters $p = 1.0$, $\Omega/J = 0.0$ and $D/J = -1.2$, $-1.4$. In this figure, solid and dashed curves correspond to the second and first order transitions respectively. For instance, if we cool the system starting from a finite temperature $T > T_c$, the system undergoes a phase transition from paramagnetic to ferromagnetic phases at a second order transition temperature $T = T_c$. If we keep on cooling process then the second order transition at a finite temperature is followed by a first order transition at a lower temperature $T < T_c$. As a last example, the unusual behavior of magnetization can be seen in the upper left panel in Fig. 8. Namely, the occurrence of three successive phase transitions in the system. For $p = 0.66$, $D/J = 0.5$ and $\Omega/J = 0.22$, $0.235$, when the system is cooled from a finite temperature, $m_z$ exhibits two successive second order transitions then it remains at a disordered state for a while and then increases continuously to a finite value at another second order transition temperature.

4. Conclusions

In this work, we have studied the phase diagrams of a bond diluted spin-1 transverse Ising model with crystal field interaction on a honeycomb lattice within the framework of an effective field approximation that takes into account the correlations between different spins in the cluster of considered lattice. We have examined the variation of the bond percolation threshold $p_c$ with the crystal and transverse field interactions which has not been reported in the literature before. In the absence of crystal and transverse fields the percolation threshold value is obtained as $p_c = 0.5397$ which improves the results obtained by other EFT based approximations.

We have also given the proper phase diagrams, especially the first order transition lines that include reentrant phase transition regions. A number of interesting and unusual phenomena such as the reentrant behavior for positive valued crystal fields and three successive phase transitions that arise for the specific range of Hamiltonian parameters $p$, $\Omega/J$ and $D/J$ have been found. As a new result, we can conclude that three successive phase transition behaviors can be observed
for the systems with bond dilution in the presence of homogeneous crystal and transverse fields, as well as the systems with random crystal fields. Hence, the results show that the type of the transition (first or second order), the existence of reentrant magnetism as well as three successive phase transitions originate from a complicated competition between bond dilution and the strength of crystal and transverse field interactions on the lattice. The behaviors shown in Figs. 1 and 5 cannot be observed by ignoring any of these Hamiltonian parameters in the system.

We hope that the results obtained in this work may be beneficial from both theoretical and experimental points of view.

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Appendix

The complete set of twenty three linear equations of the spin-1 honeycomb lattice \( q = 3 \):
\[
\langle S_1^0 \rangle_r = l_0 + 3k_1 \langle S_1 \rangle_r + 3(l_1 - l_0) \langle S_1^2 \rangle_r + 6(k_2 - k_1) \langle S_1 S_2 \rangle_r + 3(l_2 - l_1) \langle S_1 S_2^2 \rangle_r,
\]
\[
+ 3(l_0 - l_2 + l_3) \langle S_1^2 S_2 \rangle_r + k_3 \langle S_1 S_2 S_3 \rangle_r + 3(l_4 - l_2) \langle S_1 S_2^2 S_3 \rangle_r,
\]
\[
+ 3(k_1 - 2k_2 + k_4) \langle S_1 S_2^2 S_3^2 \rangle_r + (-l_0 - 3l_1 - 3l_3 + l_4) \langle S_1^2 S_2^2 S_3^2 \rangle_r,
\]
\[
\langle S_1 S_0 \rangle_r = (3l_1 - 2l_0) \langle S_1 \rangle_r + 3k_1 \langle S_1^2 \rangle_r + 3(l_0 - l_1 + l_2 + l_3) \langle S_1 S_2 \rangle_r + 6(k_1 - k_2) \langle S_1 S_2 S_3 \rangle_r + k_3 \langle S_1 S_2 S_3 \rangle_r,
\]
\[
+ (-l_0 + 3l_1 - 3l_3 - 3l_4 + l_4) \langle S_1 S_2^2 S_3 \rangle_r + 3(1 - 2k_2 + k_4) \langle S_1^2 S_2^2 S_3^2 \rangle_r,
\]
\[
\langle S_1 S_2 S_0 \rangle_r = (l_0 - 3l_1 + 3l_2 + l_3) \langle S_1 S_2 \rangle_r + 6(k_0 - k_1) \langle S_1 S_2^2 \rangle_r + (3k_1 - 6k_2 + k_3 + 3k_4) \langle S_1 S_2^2 S_3 \rangle_r,
\]
\[
\langle S_1^2 \rangle_r = a_1 \left(1 - \langle S_1^0 \rangle^2_r \right) + a_3 \langle S_1^2 \rangle_r,
\]
\[
\langle S_1 S_2 \rangle_r = a_1 \langle S_1 \rangle_r + a_2 \langle S_0 S_1 \rangle_r + (a_3 - a_1) \langle S_1 S_3^0 \rangle_r,
\]
\[
\langle S_1 S_2 S_3 \rangle_r = a_1 \langle S_1 S_3 \rangle_r + a_2 \langle S_0 S_1 S_3 \rangle_r + (a_3 - a_1) \langle S_1 S_2 S_3^0 \rangle_r,
\]
\[
\langle S_1^2 \rangle_r = b_1 \left(1 - \langle S_1^0 \rangle^2_r \right) + b_2 \langle S_1^2 \rangle_r + b_3 \langle S_1^2 \rangle_r,
\]
\[
\langle S_1 S_2 \rangle_r = b_1 \langle S_1 \rangle_r + b_2 \langle S_0 S_1 \rangle_r + (b_3 - b_1) \langle S_1 S_3^0 \rangle_r,
\]
\[
\langle S_1 S_2 S_3 \rangle_r = b_1 \langle S_1 S_3 \rangle_r + b_2 \langle S_0 S_1 S_3 \rangle_r + (b_3 - b_1) \langle S_1 S_2 S_3^0 \rangle_r,
\]
\[
\langle S_0 S_1 S_3 \rangle_r = b_3 \langle S_0 S_3 \rangle_r + b_2 \langle S_1 S_3 \rangle_r,
\]
\[
\langle S_0 S_1 S_3 \rangle_r = b_3 \langle S_0 S_3 \rangle_r + b_2 \langle S_1 S_3 \rangle_r,
\]
\[
\langle S_0 S_1 S_3 \rangle_r = b_3 \langle S_0 S_3 \rangle_r + b_2 \langle S_1 S_3 \rangle_r,
\]
\[
\langle S_1 S_2 S_3 \rangle_r = b_1 \langle S_1 S_3 \rangle_r + b_2 \langle S_0 S_1 S_3 \rangle_r + (b_3 - b_1) \langle S_1 S_2 S_3^0 \rangle_r,
\]
\[
\langle S_1 S_2 S_3 \rangle_r = b_1 \langle S_1 S_3 \rangle_r + b_2 \langle S_0 S_1 S_3 \rangle_r + (b_3 - b_1) \langle S_1 S_2 S_3^0 \rangle_r,
\]
\[
\langle S_1 S_2 S_3 \rangle_r = b_1 \langle S_1 S_3 \rangle_r + b_2 \langle S_0 S_1 S_3 \rangle_r + (b_3 - b_1) \langle S_1 S_2 S_3^0 \rangle_r,
\]
\[
\langle S_0 S_1 S_3 \rangle_r = b_3 \langle S_0 S_3 \rangle_r + b_2 \langle S_1 S_3 \rangle_r,
\]
\[
\langle S_0 S_1 S_3 \rangle_r = b_3 \langle S_0 S_3 \rangle_r + b_2 \langle S_1 S_3 \rangle_r,
\]
\[
\langle S_0 S_1 S_3 \rangle_r = b_3 \langle S_0 S_3 \rangle_r + b_2 \langle S_1 S_3 \rangle_r,
\]
\[ \langle \langle S_0^2 \rangle \rangle_r = p_0 + 3c_1 \langle \langle S_1 \rangle \rangle_r + 3(p_1 - p_0) \langle \langle S_1^2 \rangle \rangle_r + 3p_2 \langle \langle S_1 S_2 \rangle \rangle_r + (6c_2 - c_1) \langle \langle S_1 S_2^2 \rangle \rangle_r + 3(p_0 - 2p_1 + p_3) \langle \langle S_1^2 S_2 \rangle \rangle_r + c_1 \langle \langle S_1 S_2 S_3 \rangle \rangle_r + 3(p_4 - p_2) \langle \langle S_1 S_2 S_3 \rangle \rangle_r + (c_1 - 2c_2 + c_4) \langle \langle S_1 S_2 S_3 S_4 \rangle \rangle_r + (-p_0 + 3p_1 - 3p_3 + p_5) \langle \langle S_1 S_2 S_3 S_4 \rangle \rangle_r, \]

\[ \langle \langle S_0^2 \rangle \rangle_r = v_0 + 3\mu_1 \langle \langle S_1 \rangle \rangle_r + (v_1 - v_0) \langle \langle S_1^2 \rangle \rangle_r + 3v_2 \langle \langle S_1 S_2 \rangle \rangle_r + 6(\mu_2 - \mu_1) \langle \langle S_1 S_2^2 \rangle \rangle_r + 3(v_0 - 2v_1 + v_3) \langle \langle S_1^2 S_2 \rangle \rangle_r + \mu_3 \langle \langle S_1 S_2 S_3 \rangle \rangle_r + 3(v_4 - v_2) \langle \langle S_1 S_2 S_3 \rangle \rangle_r + (3\mu_1 - 2\mu_2 + \mu_4) \langle \langle S_1 S_2 S_3 S_4 \rangle \rangle_r + (-v_0 + 3v_1 - 3v_3 + v_5) \langle \langle S_1 S_2 S_3 S_4 \rangle \rangle_r. \]

(A.1)