Example Let $\sim_r$ be the equivalence relation: "$a \sim_r b$ iff $ba^{-1} \in H$" where $H$ is a subgroup of $G$.
Then let’s calculate $[a]_{\sim_r} = \{b \in G : a \sim_r b\} = ...$

Definition: (Coset) Let $H$ be a subgroup of a group $G$. The subset (Not necessarily a subgroup) $aH = \{ah|h \in H\}$ of $G$ is the left coset of $H$ containing ‘a’, while the subset $Ha = \{ha|h \in H\}$ is the right coset of $H$ containing ‘a’.

Theorem: Let $H$ be a subgroup of $G$. Then

1-) Any left coset $aH \neq H$, if $a \notin H$.

2-) Any two left cosets of $H$ are either identical or disjoint. That is either $aH = bH$ or $aH \cap bH = \emptyset$.

3-) $G$ is the union of the left cosets of $H$. That is $G = \bigcup_{a \in G} aH$.

Proof.

1-) By contradiction, suppose

2-) Suppose that $aH \cap bH \neq \emptyset$, then we will show that $aH = bH$.

Take any $ah$ from $aH$. 
Converse is your exercise.

3-Take any \( g \in G \), then \( g = ge \in gH \)

Remark. Note that for abelian groups, the cosets are \( a + H \) and \( H + a \).

**Proposition:** If \( G \) is an abelian group, for any subgroup \( H \) of \( G \) and for any \( a \in G \),
\[ a + H = H + a. \]
Proof:
Let’s take any \( a + h \) from \( a + H \) where \( h \in H \), then...

Converse is your exercise
**Lemma:** Let $G$ be a group and $H$ be a subgroup of $G$. Then if we consider a left(right) coset $aH$ of $H$ in $G$, we say

$$|aH| = |H| = |Ha|.$$ 

Proof: Let's define a 1-1 and onto map between sets $aH$ and $H$. We define $f : aH \mapsto H$ by $f(ah) = h$. 

$f$ is well defined:

$f$ is 1-1:

$f$ is onto:

**Example:** Exhibit the left (and right) cosets of the subgroup $3\mathbb{Z}$ of $\mathbb{Z}$.

Since $\mathbb{Z}$ is abelian, right cosets are same as left cosets.
Exercises

1. Find all cosets of the subgroup $4\mathbb{Z}$ of $\mathbb{Z}$.

2. Find all cosets of the subgroup $4\mathbb{Z}$ of $2\mathbb{Z}$.

3. Find the partition of $\mathbb{Z}_8$ into cosets of the subgroup $\langle 2 \rangle$.

4. Find the partition of $\mathbb{Z}_{36}$ into cosets of the subgroup $\langle 18 \rangle$. 
5. Consider $S_3$ with elements $\rho_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$, $\rho_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $\rho_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$,

$\mu_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$, $\mu_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$, $\mu_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

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Consider the cyclic subgroup $< \mu_1 >$. Find the partition into left cosets by $< \mu_1 >$. Then find the partition into right cosets by $< \mu_1 >$. 
PART 2

Theorem: (Lagrange’s Theorem) Let $H$ be a subgroup of a finite group $G$, then the order of $H$ is a divisor of the order of $G$.

Proof: Let $g_1H, g_2H, ..., g_mH$ be all distinct left cosets of $H$ in $G$(we could also do it for right cosets). Since $G$ is finite, there could be only finite number of such cosets. (Why?)

Because the cosets are equivalence classes, $g_1H \cup g_2H \cup ... \cup g_mH = G$. Since cosets are equivalence classes, they are distinct, i.e. $g_iH \cap g_jH = \emptyset$ if $i \neq j$. Therefore

$$|g_1H| + |g_2H| + ... + |g_mH| = |G|.$$ But by using the preceding Lemma, we say

$$|H| + |H| + ... + |H| = |G|,$$ that is $m.|H| = |G|$. Therefore $|H|$ divides $|G|$. 

Remark. For infinite groups, the theorem would be nonsense.

Example: $\mathbb{Z}_6$ has no subgroup of order 4.

Corollary: The order of an element of a finite group divides the order of the group.

Proof:

Corollary: Let $G$ be a group of order $m$, then for any $g \in G$, $g^m = e$.

Proof:
Example: Let’s see above corollary in $\mathbb{Z}_6$.

**Definition:** (Index) Let $H$ be a subgroup of a group $G$. The number of left cosets of $H$ in $G$ is the index $(G : H)$ of $H$ in $G$.

**Proposition:** Let $G$ be a finite group, then $(G : H)$ is finite and $(G : H) = \frac{|G|}{|H|}$.

Proof:

**Remark:** According to the Lagrange Theorem, for finite groups, orders of all subgroups divide order of the group. But can we say the converse?

That is, if $|G| = m$, then for any $n \in \mathbb{Z}^+$ with $n|m$, can we find a subgroup $H$ of $G$ such that $|H| = n$?

No! For instance, $A_4$ has no subgroup of order 6. However, for all abelian groups, the answer is Yes!. 
Math 3055 (Algebra I)
Exercises Week 4: Cosets and Lagrange Theorem

1. Every group of prime order is cyclic. (Hint: Take a cyclic subgroup and consider with Lagrange Theorem.)

2. Let $G$ be a group of order $pq$, where $p$ and $q$ are prime numbers. Show that every proper subgroup of $G$ is cyclic.

3. Find all cosets of the subgroup $<4>$ of $\mathbb{Z}_{12}$. 
4. Suppose that $H$ and $K$ are subgroups of a group $G$ such that $K \leq H \leq G$ and suppose $(H : K)$ and $(G : H)$ are both finite, then show that $(G : K) = (G : H)(H : K)$.

5. Find the index of $<3>$ in the group $\mathbb{Z}_{24}$.

6. Let $H$ be a subgroup of a group $G$ such that $g^{-1}hg \in H$ for all $g \in G$ and all $h \in H$. Show that every left coset $gH$ is the same as the right coset $Hg$.

7. Show that if $H$ is a subgroup of index 2 in a finite group $G$, then every left coset of $H$ is equivalent to right coset of $H$. 