ENERGY EQUATIONS

INTRODUCTION TO FRACTURE MECHANICS

Prof. Dr. M. Evren Toygar
Griffith’s Energy balance approach

Griffith Cracks

- First documented paper on fracture (1920) Considered as father of Fracture Mechanics
Griffith’s Energy balance approach

According to first law of thermodynamics, when a system goes from a nonequilibrium state to equilibrium, there is net degrease in energy. (Griffith applied this idea to the formation of a crack)

A crack can form or an existing crack can grow only if such a process causes the total energy to degrease or remain constant.

The change in potential energy of an infinite plate, which includes elliptical crack and subjected to Mode I loading, can be given as:

\[ U - U_0 = -\frac{\pi \sigma^2 a^2 t}{E} + 4at\gamma_s \]

- \( U = \) potential energy of body with crack
- \( U_0 = \) potential energy of body without crack
- \( \sigma = \) applied stress
- \( a = \) half of a crack length
- \( t = \) thickness
- \( E = \) modulus of elasticity
- \( \gamma_s = \) specific surface energy
Griffith’s Energy balance approach

Condition of Equilibrium:

\[ U = -\frac{\pi \sigma^2 a^2 t}{E} + 4at \gamma_s + U_0 \]

By differentiating the potential energy \( U \) with respect to crack length and equating to zero:

\[ \frac{\partial U}{\partial a} = -\frac{2\pi \sigma^2 a t}{E} + 4t \gamma_s + U_0 \]

\( \partial U_0 / \partial a = 0 \), since \( U_0 \) accounts for potential energy of the body without a crack and does not vary with crack length. Then, the equilibrium equation is obtained as,

\[ \frac{\pi \sigma^2 a}{E} = 2\gamma_s \]

If \( \partial^2 U_0 / \partial a^2 \) has negative term, it means that the equilibrium condition described is unstable and unstable crack growth will occur.
Griffith’s criterion (energy based)

\[ \left( \frac{dU}{da} \right)_{a_{cr}} = 0 \]

\[ U_y = 4 \gamma_f a \]

\[ U_s = \frac{\pi a^2 \sigma^2}{E} \]

Stable cracks

Unstable cracks

Positive slope

Negative slope

Increasing stress
Griffith Theory of Brittle Fracture

- If the change in elastic strain energy due to crack extension is higher than the energy required to create new crack surfaces, crack propagation will occur. From the equilibrium equations stress can be written as:

\[ \sigma_f = \sqrt{\frac{2 \gamma E}{\pi a(1 - \nu^2)}} \]

**Griffith Equation**

**Plane strain conditions**

**Plane stress conditions**

- \( \sigma_f \): Elastic modulus
- \( \gamma \): Specific surface energy (energy to break bonds/unit area)
- \( a \): half of a crack length
Griffith (1920), realized the significance of microcracks in reducing the fracture strength.

Griffith's work, which has since been known as the Griffith energy balance approach, and has served as a foundation for fracture mechanics, deals with the equilibrium state of an elastic, solid body, deformed by specified surface forces.

Griffith extended the theorem of minimum energy by accounting for the increase of surface energy which occurs during formation of cracks.

He assumed that the equilibrium position is one in which rupture of the solid occurs if the system is allowed to pass from an unbroken to a broken state through a process involving continuous reduction of potential energy.

In a larger scale, the Griffith flaws include joints, faults, and bedding planes.
Griffith Theory (ENERGY APPROACH)

- It is important to recognize that the Griffith relation was derived for an elastic brittle materials containing a very sharp crack. When the available energy for crack growth exceeds the material resistance, the crack expansion occurs, in other words fracture occurs.

- Griffith performed balance between two energies and derived a critical stress for propagation of a crack.

- During crack propagation elastic strain energy is released.

- Fracture creates new surfaces which raises the surface energy.

- Irwin, has come up with the concept of strain energy release rate \( G \). In linear elastic materials, the energy release rate represents the elastic (potential) energy per unit crack area required for a very small crack extension.

- The speed of energy release rate while fracture occurs is equal to the critical energy dissipation rate, which is the measure of fracture toughness.
The strain energy released on the introduction of a center crack with ‘2a’ length in an infinite plate of unit width (depth), under an uniform stress $\sigma$ is given by the formula as below

$$\Delta U = U_{\text{without crack}} - U_{\text{with crack}} = U_s = \frac{\pi a^2 \sigma^2}{E}$$

This is because the body with the crack has a lower elastic energy stored in it as compared to the body without the crack. The units of $U_s$ is [J/m].

- It represents the change in elastic strain energy of a plate which includes crack initiation especially it is a difference between two energies.
- Half crack length ‘a’ appears in the formula.
- $E$ is assumed constant in the process (the apparent modulus will decrease slightly).
- $\sigma_a$ is the ‘far field’ stress (this may result from displacements rather than from applied forces).
Elastic energy released from a plate with a center crack

\[ U_s = \frac{\pi a^2 \sigma^2}{E} \]  

(1)

\[ \frac{\partial U_s}{\partial a} = \frac{2\pi a \sigma^2}{E} \]

For a body in plane strain condition (i.e. \( \sim \) thick in the z-direction, into the plane of the page), E is replaced with \( E/(1-\nu^2) \):

\[ U_s = \frac{\pi a^2 \sigma^2}{E/(1-\nu^2)} \]  

Plane strain condition

Energy released from this circular region is given by the formula (1)

Plane stress condition

Stress state: \( \sigma_{xx} = \sigma_a \)

Prof. Dr. M. Evren TOYGAR
The fracture surface energy of the crack of “2a “ length is:

\[
\text{Fracture surface energy} = U_\gamma = 4 \gamma_f a
\]

\[
\frac{\partial U_\gamma}{\partial a} = 4 \gamma_f
\]

- Fracture surface energy occurs due to plastic deformation, micro-cracking, and the energy of the ‘broken bonds’.
- The units are Joules per meter depth of the body: [J/m].
According to Griffith’s criterion:

\[
\frac{dU_s}{da} \geq \frac{dU_\gamma}{da} \quad \Rightarrow \quad \frac{2\pi a \sigma^2}{E} \geq 4\gamma_f
\]

At criticality crack propagation just starts:

\[
\frac{\partial U_s}{\partial a} = \frac{2\pi a \sigma^2}{E} \quad \text{and} \quad \frac{\partial U_\gamma}{\partial a} = 4\gamma_f
\]

Inner outer products introduce the equation as:

\[
\sigma_f = \sqrt{\frac{2\gamma E}{\pi a_{cr}}}
\]

The minimum crack size, which will propagate with a ‘balance’ in energy.

The critical crack size \((a_{cr})\):

\[
a_{cr} = \frac{2E\gamma_f}{\pi \sigma_f^2} \quad \geq a
\]

A crack below this critical size will not propagate under a constant stress \(\sigma_a\).

A crack of size greater than or equal to \(a_{cr}\) will propagate.

This stress \(\sigma\) now becomes the fracture stress \(\sigma_f\) \(\rightarrow\) cracks of length \(a_{cr}\) will grow unstable if the stress exceeds \(\sigma_f\).
R-Curve

The effect of toughening mechanisms ahead of the crack tip is sometimes represented by a rising resistance curve or R-curve, in which the critical energy release rate, is not constant, but changes with the crack extension. As the applied stress increases, the energy release rate $G$ changes. Until the $G$ function becomes tangential to the R-curve, the crack extension is stable.

For the conditions of **stable crack growth** the following equations must be satisfied:

$G = R$

\[
\frac{\partial G}{\partial a} < \frac{\partial R}{\partial a}
\]

For the conditions of **unstable crack growth** the following equation must be checked:

\[
\frac{\partial G}{\partial a} > \frac{\partial R}{\partial a}
\]
Instability and R-Curve

R-curve For Brittle Materials

- Negligible size of plastic zone in the vicinity of crack tip
- No stable growth if \( G \geq R \)

Stable & Unstable Crack Growth

For a crack to grow & become critical

1) \( G \geq R \)

2) \( \frac{dG}{da} \geq \frac{dR}{da} \)

a) Schematic driving force vs. flat R curve diagrams

b) Schematic driving force vs. rising R curve diagrams
G is defined as the total potential energy ($\Pi$) decrease during unit crack extension ($da$). ‘G’ is also referred to as the crack extension force and is given by:

$$G = -\frac{d\Pi}{da}$$

The potential energy is a difficult quantity to visualize. In the absence of external tractions (i.e. *only displacement boundary conditions are imposed*), the potential energy is equal to the strain energy stored: $\Pi = U_s$

$$G = -\frac{dU_s}{da}$$

With *displacement boundary conditions only* ‘G’ has units of [J/m²] = [N/m]

As we have seen, $U_s$ is given by:

$$U_s = \frac{\pi a^2 \sigma_a^2}{E}$$

For perfectly brittle solids: $G_C = 2\gamma_f$ (i.e. this is equivalent to Griffith’s criterion).

Crack growth occurs if G exceeds (or at least equal to) a critical value $G_C$, which is the fracture toughness of the material.
In spite of the fact that ‘G’ has a more direct physical interpretation for the crack growth process, usually we work with ‘K’ as it is more amenable to theoretical computation.

‘K’ can be related to ‘G’ using the following equations:

**Plane stress:** \( K^2 = GE \)

**Plane strain:** \( K^2 = \frac{G.E}{1 - v^2} \)
Problems:

Problem 1: On a plate containing an edge crack, $\beta = 1.12$. $R$ curves representing the resistance of the material to fracture progression for plane stress is

$$ R = \frac{K_{IC}^2}{E} + 0.2 \times (\Delta a + 0.02)^{0.5} $$

Here, the fracture toughness is 50 MN/m$^{3/2}$ elastic modulus is $2 \times 10^5$ MN/m$^2$. $\Delta a$ is the amount of steady cracks going forward. Initial crack length is $a_i = 10$ cm

Therefore,

a) calculate stable crack growth,

b) calculate fracture stress. $a_i$ is the initial crack length, $a_c$ is the final crack length

Answer 1: a) $a_c = 0.13$ m, b) $\sigma_{fr} = 150.65$ MPa
Problem 2: 150 MPa stress is applied to a plate. The fracture toughness of the material is given as 80 MPa√m. According to this,

a) calculate the critical length of the center crack in the plate,

b) calculate the energy release rate.  Note: E = 207 GPa’ dir.

Problem 3: If the specific surface energy for Polmethyl acrylate is 0.0365 J / m² and its corresponding modulus of elasticity is 2.38 GPa, compute the critical tensile stress required for unstable propagation of a central internal crack whose length is 30 mm. If the strength of the sound glass is 70 MPa, calculate the reduction in strength due to the presence of the crack.

Answer 3: 13.29%
Problem 4: A rectangular perspex plate 600 mm by 300 mm by 6 mm thick is scribed into two equal squares by a knife, leaving a uniform cut of depth 0.3 mm. What is the bending moment required to break the plate if the perspex has a work to fracture of 500 J/m²? Note that $E = 2.5$ GPa for perspex.

![Diagram of a rectangular plate with a uniform cut](image)

\[
M = \frac{\sigma_f I}{\tfrac{1}{2}(t - a)} \quad \text{where } t = \text{thickness and } I = \tfrac{1}{12} b(t - a)^3 \quad \sigma_f = \left[ \frac{G_c E}{\pi a} \right]^{\frac{1}{2}}
\]

\[
\sigma_f = \left[ \frac{500 \times 2.5 \times 10^5}{\pi \times 0.0003} \right]^{\frac{1}{2}} \quad \text{i.e. } \sigma_f = 36.4 \text{ MPa}
\]

\[
M = \frac{36.4 \times 10^5 \times 0.3 \times 0.0057^3}{12 \times 0.00285} = 59.2 \text{ Nm}
\]
**Problem 5:** A polystyrene component must not fail when a tensile stress of 1.25 MPa (180 psi) is applied. Determine the maximum allowable surface crack length if the surface energy of polystyrene is 0.50 J/m². Assume a modulus of elasticity of 3.0 GPa.

**Problem 6:** A polystyrene component must not fail when a tensile stress of 1.25 MPa (180 psi) is applied. Determine the maximum allowable surface crack length if the surface energy of polystyrene is 0.50 J/m². Assume a modulus of elasticity of 3.0 GPa.

**Answer 6:** \[ a = 5: 6.1 \times 10^{-4} \text{m} = 0.61 \text{ mm (0.024in)}. \]