THE INDUCTIVE CLOSURE OF SUPPLEMENTS

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Abstract

We deal with two proper classes defined by means of complements (closed submodules) and supplements in modules and their relations with the neat and coneat short exact sequences of modules. For a Dedekind domain \( W \), a finitely generated torsion submodule of a \( W \)-module is a supplement if and only if it is a complement. The inductive closure of the proper class \( \text{Suppl}_{\text{Z-Mod}} \) is flatly generated by all simple abelian groups \( (\mathbb{Z}/p\mathbb{Z}, p \) prime), so it is equal to \( \text{Compl}_{\text{Z-Mod}} = \text{Neat}_{\text{Z-Mod}} \). The functor \( \text{Ext}_{\text{Suppl}_{\text{Z-Mod}}} \) is not factorizable as \( R\text{-Mod} \times R\text{-Mod} \xrightarrow{\text{Ext}} \text{Ab} \xrightarrow{H} \text{Ab} \) for any functor \( H : \text{Ab} \longrightarrow \text{Ab} \) on the category of abelian groups.

Key words: Complement, supplement, closed submodule, proper class, inductive closure of a proper class, flatly generated proper class, injectively generated proper class, projectively generated proper class, factorizable Ext.

1 Introduction

Throughout this article, by a ring we shall mean an associative ring with unity; \( R \) will denote such a general ring, unless otherwise stated. We shall consider unital left \( R \)-modules; \( R \)-module will mean left \( R \)-module. \( R\text{-Mod} \) denotes the category of all left \( R \)-modules. \( \mathbb{Z} \) denotes the ring of integers. Group will mean abelian group only. By \( W \), we denote a commutative Dedekind domain. All definitions not given here can be found in [1], [2] and [3].

A submodule \( A \) of a module \( B \) is said to have a complement in \( B \) if there exists a submodule \( K \) of \( B \) maximal with respect to \( K \cap A = 0 \). A submodule \( A \) of a module \( B \) is said to be a complement in \( B \) if \( A \) is a complement of some submodule of \( B \). It is said that \( A \) is closed in \( B \) if \( A \) has no proper essential extension in \( B \) and it is known that closed submodules and complements in a module coincide (see §1 in [1]).

Dually, a submodule \( A \) of a module \( B \) is said to have a supplement in \( B \) if there exits a submodule \( K \) of \( B \) minimal with respect to \( K + A = M \); equivalently \( K + A = M \) and \( K \cap A \) is small\((=superfluous) \) in \( K \) (which is denoted by \( K \cap A \ll K \), meaning that for no proper submodule \( X \) of \( K \), \( K \cap A + X = K \)). A submodule \( A \) of a module \( B \) is said to be a supplement in \( B \) if \( A \) is a supplement of some submodule of \( B \). Unlike complements a submodule of a module may

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not have any supplements. If every submodule of a module has a supplement, then it is said to be a supplemented module. For the definitions and related properties see §41 in [2].

We deal with complements (closed submodules) and supplements in unital $R$-modules for an associative ring $R$ with unity using relative homological algebra via the known two dual proper classes of short exact sequences of $R$-modules and $R$-module homomorphisms, $\text{Compl}_{R\text{-Mod}}$ and $\text{Suppl}_{R\text{-Mod}}$, and related other proper classes like $\text{Neat}_{R\text{-Mod}}$ and $\text{CoNeat}_{R\text{-Mod}}$. $\text{Compl}_{R\text{-Mod}} [\text{Suppl}_{R\text{-Mod}}]$ consists of all short exact sequences

![Diagram]

of $R$-modules and $R$-module homomorphisms such that $\text{Im}(f)$ is a complement [resp. supplement] in $B$. $\text{Neat}_{R\text{-Mod}} [\text{CoNeat}_{R\text{-Mod}}]$ consists of all short exact sequences of $R$-modules and $R$-module homomorphisms with respect to which every simple module is projective [resp. every module with zero radical is injective].

In Section 3, the relations between the proper classes $\text{Compl}_{R\text{-Mod}}$, $\text{Suppl}_{R\text{-Mod}}$, $\text{Neat}_{R\text{-Mod}}$ and $\text{CoNeat}_{R\text{-Mod}}$ are summarized where the terminology and notation for proper classes are given in the next section.

In Section 4, we investigate some properties of these proper classes when $R = \mathbb{Z}$, the ring of integers, and when $R$ is a Dedekind domain. The inductive closure of the proper class $\text{Suppl}_{\mathbb{Z}\text{-Mod}}$ is shown to coincide with $\text{Compl}_{\mathbb{Z}\text{-Mod}} = \text{Neat}_{\mathbb{Z}\text{-Mod}}$ in Section 5.

To every proper class $\mathcal{A}$, we have a relative $\text{Ext}_\mathcal{A}$ functor and for the proper class $\text{Suppl}_{\mathbb{Z}\text{-Mod}}$, this functor behaves badly in the sense explained in Section 6.

## 2 Terminology and notation for proper classes

Let $\mathcal{A}$ be a class of short exact sequences of $R$-modules and $R$-module homomorphisms. If a short exact sequence

![Diagram]

belongs to $\mathcal{A}$, then $f$ is said to be an $\mathcal{A}$-monomorphism and $g$ is said to be an $\mathcal{A}$-epimorphism (both are said to be $\mathcal{A}$-proper and the short exact sequence is said to be an $\mathcal{A}$-proper short exact sequence.). The class $\mathcal{A}$ is said to be proper if it satisfies the following conditions (see Ch. 12, §4 in [4] or [5] or [6]):

1. If a short exact sequence $E$ is in $\mathcal{A}$, then $\mathcal{A}$ contains every short exact sequence isomorphic to $E$.
2. $\mathcal{A}$ contains all splitting short exact sequences.
3. The composite of two $\mathcal{A}$-monomorphisms is an $\mathcal{A}$-monomorphism if this composite is defined. The composite of two $\mathcal{A}$-epimorphisms is an $\mathcal{A}$-epimorphism if this composite is defined.
4. If $g$ and $f$ are monomorphisms, and $g \circ f$ is an $\mathcal{A}$-monomorphism, then $f$ is an $\mathcal{A}$-monomorphism. If $g$ and $f$ are epimorphisms, and $g \circ f$ is an $\mathcal{A}$-epimorphism, then $g$ is an $\mathcal{A}$-epimorphism.
An important example for proper classes in abelian groups is \( \text{Pure}_{\mathbb{Z}, \text{Mod}} \): The proper class of all short exact sequences (1) of abelian groups and abelian group homomorphisms such that \( \text{Im}(f) \) is a pure subgroup of \( B \), where a subgroup \( A \) of a group \( B \) is pure in \( B \) if \( A \cap nB = nA \) for all integers \( n \) (see §26-30 in [7] for purity in abelian groups). The proper class \( \text{Pure}_{\mathbb{Z}, \text{Mod}} \) forms one of the origins of relative homological algebra; it is the reason why proper classes are also called purities (as in [8], [9], [10], [11]).

Denote by \( \mathcal{A} \) a proper class of \( \mathcal{R} \)-modules. An \( \mathcal{R} \)-module \( M \) is said to be \( \mathcal{A} \)-projective \([\mathcal{A} \text{-injective, } \mathcal{A} \text{-flat}] \) if it is projective \([\text{resp. injective, flat}] \) with respect to all short exact sequences in \( \mathcal{A} \), that is, \( \text{Hom}(M, \mathcal{E}) \) \([\text{resp. } \text{Hom} (\mathcal{E}, M), M \otimes \mathcal{E}] \) is exact for every \( \mathcal{E} \) in \( \mathcal{A} \). Denote all \( \mathcal{A} \)-projective \([\mathcal{A} \text{-injective, } \mathcal{A} \text{-flat}] \) modules by \( \pi(\mathcal{A}) \) \([\text{resp. } \iota(\mathcal{A}), \tau(\mathcal{A})] \). For a given class \( \mathcal{M} \) of modules, denote by \( \pi^{-1}(\mathcal{M}) \) \([\iota^{-1}(\mathcal{M}), \tau^{-1}(\mathcal{M})] \), the smallest proper class \( \mathcal{A} \) for which each \( M \in \mathcal{M} \) is \( \mathcal{A} \)-projective \([\text{resp. } \mathcal{A} \text{-injective, } \mathcal{A} \text{-flat}] \); it is called the proper class projectively generated \([\text{resp. injectively generated, flatly generated}] \) by \( \mathcal{M} \). See §1-3,8 in [6] for these concepts in relative homological algebra in categories of modules.

For a proper class \( \mathcal{A} \) and \( \mathcal{R} \)-modules \( A, C \), denote by \( \text{Ext}_1^{\mathcal{A}}(C, A) \) or just by \( \text{Ext}_A(C, A) \), the equivalence classes of all short exact sequences in \( \mathcal{A} \) which start with \( A \) and end with \( C \), i.e. a short exact sequence in \( \mathcal{A} \) of the form (1). This turns out to be a subgroup of \( \text{Ext}_R(C, A) \) and a bifunctor \( \text{Ext}_1^{\mathcal{A}} : \mathcal{R} \text{-Mod} \times \mathcal{R} \text{-Mod} \rightarrow \mathcal{A} \text{-Mod} \) is obtained which is a subfunctor of \( \text{Ext}_1^R \) (see Ch. 12, ¶4-5 in [4]).

A proper class \( \mathcal{A} \) is said to be inductively closed if for every direct system \( \{ \mathcal{E}_i (i \in I) ; \pi_i (i \leq j) \} \) in \( \mathcal{A} \), the direct limit \( \mathcal{E} = \lim \mathcal{E}_i \) is also in \( \mathcal{A} \) (see [12] and §8 in [6]). As in [12], for a proper class \( \mathcal{A} \), denote by \( \overline{\mathcal{A}} \), the smallest inductively closed proper class containing \( \mathcal{A} \); it is called the inductive closure of \( \mathcal{A} \).

## 3 The proper classes \( \text{Suppl}_{\mathcal{R} \text{-Mod}}, \text{Compl}_{\mathcal{R} \text{-Mod}}, \text{Neat}_{\mathcal{R} \text{-Mod}} \) and \( \text{Co-Neat}_{\mathcal{R} \text{-Mod}} \) for a ring \( \mathcal{R} \)

The classes \( \text{Compl}_{\mathcal{R} \text{-Mod}} \) and \( \text{Suppl}_{\mathcal{R} \text{-Mod}} \) defined in the first section really form proper classes as has been shown more generally at Theorem 1 in [10], at Theorem 1 in [11], at Proposition 4 and Remark after Proposition 6 in [13]. In [13], following the terminology in abelian groups, the term ‘high’ is used instead of complements and ‘low’ for supplements. [10] and [11] use the terminology ‘high’ and ‘cohigh’ for complements and supplements, and give more general definitions for proper classes of complements and supplements related with another given proper class (motivated by the considerations as pure-high extensions and neat-high extensions in [14]); ‘weak purity’ in [10] is what we denote by \( \text{Compl}_{\mathcal{R} \text{-Mod}} \).

A subgroup \( A \) of a group \( B \) is said to be neat in \( B \) if \( A \cap pB = pA \) for all prime numbers \( p \) (see §31 in [7]; the notion of neat subgroup has been introduced in [15]). Denote by \( \text{Neat}_{\mathbb{Z}, \text{Mod}} \), the proper class of all short exact sequences (1) of abelian groups and abelian group homomorphisms where \( \text{Im}(f) \) is a neat subgroup of \( B \). The following result is one of the motivations for us to deal with complements and its dual supplements:

**Theorem 3.1.** The proper class \( \text{Compl}_{\mathbb{Z}, \text{Mod}} = \text{Neat}_{\mathbb{Z}, \text{Mod}} \) is projectively generated, flatly generated and injectively generated by simple groups \( \mathbb{Z}/p\mathbb{Z}, p \) prime...
\[ \text{Compl}_{\mathbb{Z} \text{-Mod}} = \text{Neat}_{\mathbb{Z} \text{-Mod}} = \pi^{-1}(\{\mathbb{Z}/p\mathbb{Z} \mid p \text{ prime}\}) = \tau^{-1}(\{\mathbb{Z}/p\mathbb{Z} \mid p \text{ prime}\}) = \iota^{-1}(\{\mathbb{Z}/p\mathbb{Z} \mid p \text{ prime}\}). \]

**Proof.** The first equality has been pointed out after Definition 3 in §4 of [14]; more generally, the equality of the first and third proper classes above will follow from Corollary to Proposition 8 in [13] or Theorem 5 in [10]. The second and third equalities follow from Proposition 2 in [16], since this proposition gives for the ring \( \mathbb{Z} \) and prime number \( p \) that for a short exact sequence (1) of abelian groups and homomorphisms, call it \( E \), Hom(\( \mathbb{Z}/p\mathbb{Z}, E \)) is exact if and only if \( \mathbb{Z}/p\mathbb{Z} \otimes E \) is exact if and only if \( \text{Im}(f) \cap pB = p\text{Im}(f) \).

So, it remains only to prove \( \text{Neat}_{\mathbb{Z} \text{-Mod}} = \iota^{-1}(\{\mathbb{Z}/p\mathbb{Z} \mid p \text{ prime}\}) \). \((\subseteq)\) part follows from Lemma 4 in [14] (as it implies in particular that every simple group \( \mathbb{Z}/p\mathbb{Z} \) (\( p \) prime), is \( \text{Neat}_{\mathbb{Z} \text{-Mod}} \)-injective); as proved there, this follows because if a simple group \( \mathbb{Z}/p\mathbb{Z} \) is neat in a group, then it is pure there, so being of bounded order, it is a direct summand. The proof of \((\supseteq)\) part (and also \((\subseteq)\) part) is done in Theorem 7 in [17]. \(\square\)

The second equality in the previous theorem motivates one to define for \( R \)-modules,
\[ \text{Neat}_{R \text{-Mod}} \overset{\text{def.}}{=} \pi^{-1}(\{\text{all simple } R\text{-modules}\}) = \pi^{-1}(\{R/P \mid P \text{ maximal left ideal of } R\}), \]
following 9.6 in §9 of [5] (and [13]).

**Proposition 3.2.** (Proposition 5 in [13]) \( \text{Compl}_{R \text{-Mod}} \subseteq \text{Neat}_{R \text{-Mod}} \) for any ring \( R \).

**Proposition 3.3.** (Corollary to Proposition 8 in [13]) If \( R \) is a commutative Noetherian ring in which every nonzero prime ideal is maximal, then \( \text{Compl}_{R \text{-Mod}} = \text{Neat}_{R \text{-Mod}} \).

[10] gives a characterization of this equality in terms of the ring \( R \). A ring \( R \) is said to be a \( C \)-ring if for every essential left ideal \( I \) of \( R \), there exist \( r \in R \) such that \( (I : r) = \{s \in R \mid sr \in I\} \) is a maximal left ideal of \( R \). For example a commutative Noetherian ring in which every prime ideal \( \neq 0 \) is maximal is a \( C \)-ring (using primary decomposition). So, of course, in particular a Dedekind domain or a principal ideal domain (shortly PID) is also a \( C \)-ring. The previous proposition is then a corollary of this:

**Theorem 3.4.** (by Theorem 5 in [10]) For a ring \( R \), \( \text{Compl}_{R \text{-Mod}} = \text{Neat}_{R \text{-Mod}} \) if and only if \( R \) is a \( C \)-ring.

We have,
\[ \text{Neat}_{R \text{-Mod}} = \pi^{-1}(\{\text{all semisimple } R\text{-modules}\}) = \pi^{-1}(\{M \mid \text{Soc } M = M, M \text{ an } R\text{-module}\}), \]
where \( \text{Soc } M \) is the socle of \( M \), that is the sum of all simple submodules of \( M \). Dualizing this, we define the proper class \( \text{CoNeat}_{R \text{-Mod}} \) by
\[ \text{CoNeat}_{R \text{-Mod}} = \iota^{-1}(\{\text{all } R\text{-modules with zero radical}\}) = \iota^{-1}(\{M \mid \text{Rad } M = 0, M \text{ an } R\text{-module}\}). \]
Proposition 3.5. For any ring \( R \),

\[
\text{Suppl}_{R\text{-Mod}} \subseteq \text{CoN eat}_{R\text{-Mod}} \subseteq \iota^{-1}(\{ \text{all (semi-)simple } R\text{-modules} \})
\]

Proof. Let \( M \) be an \( R \)-module such that \( \text{Rad} \ M = 0 \). To prove the first inclusion it suffices to show that any short exact sequence

\[
E: \quad 0 \rightarrow M \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0
\]

in \( \text{Suppl}_{R\text{-Mod}} \) splits. Since \( E \in \text{Suppl}_{R\text{-Mod}} \), \( \text{Im}(f) \), call it \( M' \), is a supplement in \( B \); so there exists a submodule \( K \) of \( B \) such that \( M' \) is a supplement of \( K \) in \( B \), hence

\[
M' + K = B \text{ and } M' \cap K \ll M'
\]

As \( M' \cap K \ll M' \), \( M' \cap K \subseteq \text{Rad} \ M' \cong \text{Rad} \ M = 0 \), so \( M' \cap K = 0 \). Thus \( B = M' \oplus K \), hence the sequence \( E \) splits.

The last inclusion holds since for a semisimple module \( M \), \( \text{Rad} \ M = 0 \). \( \square \)

[11] gives the following interesting result (the equality from Theorem 5 in [10]):

Theorem 3.6. (Corollary 1 and 6 in [11]) For a Dedekind domain \( W \),

\[
\text{Suppl}_{W\text{-Mod}} \subseteq \text{Compl}_{W\text{-Mod}} = \text{Neat}_{W\text{-Mod}}
\]

where the inclusion is strict if \( W \) is not a field. So if \( A \) is a supplement in an \( W \)-module \( B \) where \( W \) is a Dedekind domain, then \( A \) is a complement.

As in abelian groups (Theorem 3.1), \( \text{Compl}_{W\text{-Mod}} \) is both projectively generated, injectively generated and flatly generated:

Theorem 3.7. The following five proper classes of \( W \)-modules are equal for a Dedekind domain \( W \):

1. \( \text{Compl}_{W\text{-Mod}} \),
2. \( \text{Neat}_{W\text{-Mod}} \overset{\text{df}}{=} \pi^{-1}(\{ W/P | P \text{ maximal ideal of } W \}) \),
3. \( \iota^{-1}(\{ M | M \in W\text{-Mod} \text{ and } PM = 0 \text{ for some maximal ideal } P \text{ of } W \}) \),
4. \( \tau^{-1}(\{ W/P | P \text{ maximal ideal of } W \}) \)
5. The proper class of all short exact sequences

\[
E: \quad 0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0
\]

of \( W \)-modules and \( W \)-module homomorphisms such that for every maximal ideal \( P \) of \( W \),

\[
A' \cap PB = PA', \quad \text{where } A' = \text{Im}(f)
\]

(or \( A \cap PB = PA \) when \( A \) is identified with its image and \( f \) is taken as the inclusion homomorphism).
Proof. The equality of the first two classes follows by Proposition 3.3 or Theorem 3.4.

The equality of the last two proper classes is by Lemma 6.1 in [6].

The equality of the second and fifth proper classes follows from Lemma 4.4 and Lemma 5.2 in [18].

The equality of the third and fifth proper classes follows from Theorem 5.1 in [18] as pointed out in the proof of Corollary 1 in [11]: Take a short exact sequence $E$ in the third proper class above, of the form (2) where without loss of generality $A$ is identified with its image in $B$ and $f$ is taken as the inclusion homomorphism. For a maximal ideal $P$ in $W$, since $A/PA = 0$, it is injective with respect to $E$, so it is trivial over $A/PA$ in the sense defined before Theorem 5.1 in [18]; hence by that theorem, $A \cap PB = PA$. So $E$ is in the last proper class. Conversely, to show that the last proper class is contained in the third proper class, it suffices to show that any short exact sequence $E$ in the last proper class above, of the form (2), where without loss of generality $A$ is identified with its image in $B$ and $f$ is taken as the inclusion homomorphism, is splitting if $A$ is a module such that $P A = 0$ for some maximal ideal $P$ of $W$. By Theorem 5.1 in [18], as $A \cap PB = PA$, $E$ is trivial over $A/PA = A/0 \cong A$, which means the existence of a map $\psi : B \rightarrow A$ such that $\psi(a) = a$ for all $a \in A$, hence $E$ is splitting as required.

Another consequence of Theorem 5.1 in [18] is:

**Theorem 3.8.** For a Dedekind domain $W$, and $W$-modules $A, C$,

$$\text{Ext}_{\text{Compl} W \text{-Mod}}(C, A) = \text{Ext}_{\text{Neat} W \text{-Mod}}(C, A) = \text{Rad}(\text{Ext}_W(C, A)).$$

Proof. $\text{Compl} W \text{-Mod} = \text{Neat} W \text{-Mod}$ by the previous Theorem 3.7. Take a short exact sequence $E$ in $\text{Compl} W \text{-Mod}$, of the form (2) where without loss of generality $A$ is identified with its image in $B$ and $f$ is taken as the inclusion homomorphism. For each maximal ideal $P$ in $W$, by the previous Theorem 3.7, $A \cap PB = PA$, hence by Theorem 5.1 in [18], the equivalence class $[E]$ of the short exact sequence $E$ is in $P \text{Ext}(C, A)$. Thus $[E] \in \bigcap_{P \leq \text{maximal}} P \text{Ext}(C, A) = \text{Rad}(\text{Ext}(C, A))$ (where the last equality follows from, for example, Lemma 3 in [11]).

**Corollary 3.9.** (Exercise 53.4 in [7]) For the proper class $C = \text{Compl}_Z \text{-Mod} = \text{Neat}_Z \text{-Mod}$,

$$\text{Ext}_C(C, A) = \bigcap_{p \text{ prime}} p \text{Ext}(C, A).$$

When considering complements in modules with finite uniform dimension (Goldie dimension), we have the following criterion:

**Theorem 3.10.** ((5.10 (1)) in [1]) Let $B$ be a module with finite uniform dimension (Goldie dimension) (denoted by $\text{u. dim } B < \infty$). Then a submodule $A$ of $B$ is a complement in $B$ (closed in $B$) if and only if $A$ and $B/A$ have finite uniform dimension and

$$\text{u. dim } B = \text{u. dim } A + \text{u. dim } (B/A).$$
In a finitely generated abelian group, we can look into the complements using this criterion. A finitely generated abelian group, being a direct sum of cyclic groups of infinite or prime power order, has finite uniform dimension which is the number of summands in such a direct sum decomposition because $\text{u.dim } \mathbb{Z} = 1 = \text{u.dim } \mathbb{Z}/p^k\mathbb{Z}$ for a prime number $p$ and $k \in \mathbb{Z}^+$, and u.dim is additive on finite direct sums. So a subgroup $A$ of a finitely generated abelian group is a complement if and only if $\text{u.dim } B = \text{u.dim } A + \text{u.dim } B/A$, or in terms of short exact sequences:

**Corollary 3.11.** For a finitely generated abelian group $B$, a short exact sequence

\[
0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0
\]

of abelian groups and homomorphisms is in $\text{Compl}_{\mathbb{Z}\text{-Mod}} = \text{Neat}_{\mathbb{Z}\text{-Mod}}$ if and only if

$$\text{u.dim } B = \text{u.dim } A + \text{u.dim } C.$$  

Dually for supplements in modules with finite hollow dimension, we have:

**Theorem 3.12.** (Corollary 3.2.3 in [19]) Let $B$ be a module with finite hollow dimension (dual Goldie dimension) (denoted by $h.\dim B < \infty$). Then a submodule $A$ of $B$ is a supplement in $B$ if and only if $A$ and $B/A$ have finite hollow dimension and

$$h.\dim B = h.\dim A + h.\dim(B/A).$$

For cyclic groups of prime power order, $h.\dim \mathbb{Z}/p^k\mathbb{Z} = 1$ for $p$ prime and $k \in \mathbb{Z}^+$, but $h.\dim \mathbb{Z} = \infty$. $\mathbb{Z}/p^k\mathbb{Z}$, being a uniserial $\mathbb{Z}$-module, is both uniform and hollow but $\mathbb{Z}$, although uniform, does not have finite hollow dimension. Among finitely generated abelian groups only the torsion ones (i.e. finite groups) has finite hollow dimension which is the number of summands in its decomposition as a direct sum of cyclic groups of prime power order. So:

**Corollary 3.13.** For a finite abelian group $B$, a short exact sequence

\[
0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0
\]

of abelian groups and homomorphisms is in $\text{Suppl}_{\mathbb{Z}\text{-Mod}}$ if and only if

$$h.\dim B = h.\dim A + h.\dim C.$$  

4 Finitely generated torsion complement submodules are supplements in modules over Dedekind domains

We look for some converse results to Theorem 3.6, that is when is a complement also a supplement? We shall give an example of a complement in a finitely generated abelian group which is not a supplement. We shall see that finite subgroups of a group which are complements in that group are also supplements.

Firstly, let us take a finite nonzero $p$-group $A$, where $p$ is a prime number. By the fundamental theorem for the structure of the finitely generated abelian
groups, we know that $A$ is isomorphic to a finite direct sum of cyclic groups of order a power of $p$: For some positive integers $n, k_1, k_2, \ldots, k_n$,

$$A \cong \bigoplus_{i=1}^{n} \mathbb{Z}/p^{k_i}\mathbb{Z}$$

Then considered as a $\mathbb{Z}$-module, $\text{Rad } A$ is the Frattini subgroup of $A$ and as $A$ is a $p$-group, $\text{Rad } A = pA$. So

$$A/\text{Rad } A \cong \bigoplus_{i=1}^{n} (\mathbb{Z}/p^{k_i}\mathbb{Z})/(p\mathbb{Z}/p^{k_i}\mathbb{Z}) \cong \bigoplus_{i=1}^{n} \mathbb{Z}/p\mathbb{Z}$$

is a finite direct sum of simple modules. Suppose $A$ is a neat subgroup of a group $B$, so we have a neat monomorphism (inclusion map) $0 \rightarrow A \rightarrow B$, i.e. the short exact sequence

$$\mathbb{E} : 0 \rightarrow A \rightarrow B \rightarrow B/A \rightarrow 0$$

is in $\text{Compl}_{\mathbb{Z}\text{-Mod}} = \text{Neat}_{\mathbb{Z}\text{-Mod}}$. Since $A/\text{Rad } A$ is a finite direct sum of simple groups, it is injective with respect to the short exact sequence $\mathbb{E}$ by Theorem 3.1. Thus there exists a homomorphism $h : B \rightarrow A/\text{Rad } A$ such that $h \circ i = \sigma$, that is, $h(a) = a + \text{Rad } A$ for all $a \in A$:

$$\begin{array}{ccccccccc}
0 & \rightarrow & A & \rightarrow & B & \rightarrow & B/A & \rightarrow & 0 \\
& \uparrow & & \sigma & & \downarrow & h & & \\
& & A/\text{Rad } A & & & & & & \\
\end{array}$$

Here $i$ is the inclusion map and $\sigma$ is the canonical epimorphism. So this $h$ induces a homomorphism $h' : B/\text{Rad } A \rightarrow A/\text{Rad } A$ such that $h'(a + \text{Rad } A) = a + \text{Rad } A$, hence the short exact sequence

$$\begin{array}{ccccccc}
0 & \rightarrow & A/\text{Rad } A & \rightarrow & B/\text{Rad } A & \rightarrow & B/A & \rightarrow & 0 \\
& & & \downarrow & & \uparrow & & & & \\
& & & h' & & & & & & \\
\end{array}$$

splits and we get for some subgroup $A'$ of $B$ containing $\text{Rad } A$

$$B/\text{Rad } A = A/\text{Rad } A \oplus A'/\text{Rad } A.$$

So $A + A' = B$ and $A \cap A' \leq \text{Rad } A$. As $A \cong \bigoplus_{i=1}^{n} \mathbb{Z}/p^{k_i}\mathbb{Z}$, $\text{Rad } A = pA \cong \bigoplus_{i=1}^{n} p\mathbb{Z}/p^{k_i}\mathbb{Z}$ and as for $i = 1, 2, \ldots, n$, $p\mathbb{Z}/p^{k_i}\mathbb{Z} \ll \mathbb{Z}/p^{k_i}\mathbb{Z}$, we obtain $\text{Rad } A \ll A$. Hence $A \cap A' \ll A$, so $A$ is a supplement of $A'$ in $B$. The above argument gives that if a complement in an abelian group is a finite $p$-group, then it is a supplement. Of course, the same argument works for any finite group. More generally for finitely generated torsion modules over a Dedekind domain:

**Theorem 4.1.** Let $W$ be a Dedekind domain. Take a $W$-module $B$ and a submodule $A \leq B$. Suppose $A$ is a finitely generated torsion $W$-module. Then $A$ is a complement in $B$ if and only if $A$ is a supplement in $B$.

**Proof.** ($\Rightarrow$) already follows from Theorem 3.6. Conversely suppose $A \neq 0$ is a complement in $B$. The proof goes as the same with the arguments given for a $p$-group before the theorem. We only need to show the following:
1. $A/\text{Rad} A$ is a finitely generated semisimple $W$-module, so by Theorem 3.7, it is injective with respect to the inclusion map $A \rightarrow B$ which is a $\text{Compl}_{W\text{-Mod}}$-monomorphism.

2. $\text{Rad} A \ll A$.

Consider any nonzero ideal $I$ in $W$. Since $W$ is a Dedekind domain, $I = P_1^{r_1} \cdots P_k^{r_k}$ for distinct maximal ideals $P_1, \ldots, P_k$ in $W$ and $r_1, \ldots, r_k \in \mathbb{Z}^+$. Then $W/I \cong \bigoplus_{i=1}^{k} (W/P_i^{r_i})$ as $P_1^{r_1}, \ldots, P_k^{r_k}$ are pairwise comaximal ideals. So

$$\text{Rad}(W/I) \cong \bigoplus_{i=1}^{k} \text{Rad}(W/P_i^{r_i}) = \bigoplus_{i=1}^{k} P_i(W/P_i^{r_i}) \ll \bigoplus_{i=1}^{k} (W/P_i^{r_i}) \cong (W/I)$$

since $W/P_i^{r_i}$ is a uniserial $W$-module because for a maximal ideal $P$ and $r \in \mathbb{Z}^+$, the $W$-submodules of $W/P^r$ form the following chain:

$$0 < P^{r-1}(W/P^r) < \cdots < P^2(W/P^r) < P(W/P^r) < W/P^r,$$

(so $\text{Rad}(W/P^r) = P(W/P^r) \ll W/P^r$). Since

$$(W/P^r)/\text{Rad}(W/P^r) \cong (W/P^r)/(P(W/P^r))$$

is annihilated by $P$, it is isomorphic to a direct sum of simple modules of the form $W/P$, which are finite as $W$ is Noetherian (being a Dedekind domain). Hence

$$(W/I)/\text{Rad}(W/I) \cong \bigoplus_{i=1}^{k} (W/P_i^{r_i})/\text{Rad}(W/P_i^{r_i})$$

is also a finitely generated semisimple $W$-module.

Turning back to $A$, as $A$ is a finitely generated torsion $W$-module, it is a direct sum of finitely many cyclic $W$-modules by Theorem 10.6.8 in [20]: for some $n \in \mathbb{Z}^+$ and nonzero ideals $I_1, \ldots, I_n$ in $W$,

$$A \cong (W/I_1) \oplus \cdots \oplus (W/I_n).$$

So,

$$\text{Rad} A \cong \bigoplus_{i=1}^{n} \text{Rad}(W/I_i) \ll \bigoplus_{i=1}^{n} W/I_i \cong A$$

and

$$A/\text{Rad} A \cong \bigoplus_{i=1}^{n} (W/I_i)/\text{Rad}(W/I_i)).$$

Hence $A/\text{Rad} A$ is also a finitely generated semisimple $W$-module.

\[\square\]

**Corollary 4.2.** Let $A$ be a finite subgroup of an abelian group $B$. Then $A$ is a complement in $B$ if and only if it is a supplement in $B$. 

Generalizing Corollary 3.11 and Corollary 3.13 for modules over Dedekind domains using uniform dimension and hollow dimension, we obtain a weaker form of the previous theorem. However its proof might suggest another way for a generalization over some other class of rings:

**Theorem 4.3.** Let $W$ be a Dedekind domain, $B$ be a finitely generated torsion $W$-module and $A$ a submodule of $B$. Then $A$ is a complement in $B$ if and only if $A$ is a supplement in $B$.

**Proof.** From the proof of the previous theorem, it is seen that a finitely generated torsion $W$-module is a direct sum of finitely many $W$-modules of the form $W/P^r$ ($P$ a maximal ideal) which are both hollow and uniform, so have uniform dimension 1 and hollow dimension 1. As these dimensions are additive on finite direct sums ((5.8 (2)) in [1] and (3.1.10 (1)) in [19]), we see that a finitely generated torsion $W$-module has the same finite hollow dimension and uniform dimension. So the result follows from Theorems 3.10 and 3.12 as submodules and quotient modules of a finitely generated torsion $W$-module are also finitely generated torsion $W$-modules. 

We can not generalize this result to include finitely generated modules which are not torsion even in abelian groups as the following example shows. Also this gives a proof of the strict inclusion in Theorem 3.6 for the case $W = \mathbb{Z}$ by an example.

**Example 4.4.** The following short exact sequence of abelian groups is in $\text{Compl}_{\mathbb{Z}\text{-Mod}} = \text{Neat}_{\mathbb{Z}\text{-Mod}}$ but it is not in $\text{Suppl}_{\mathbb{Z}\text{-Mod}}$:

$$
\begin{array}{c}
\text{E: } 0 \rightarrow \mathbb{Z} \xrightarrow{f} (\mathbb{Z}/p\mathbb{Z}) \oplus \mathbb{Z} \xrightarrow{g} \mathbb{Z}/p^2\mathbb{Z} \rightarrow 0
\end{array}
$$

where $f(k) = k \cdot (-1 + p\mathbb{Z}, p)$, $k \in \mathbb{Z}$, and $g(a + p\mathbb{Z}, b) = (pa + b) + p^2\mathbb{Z}$, $a, b \in \mathbb{Z}$.

Hence this example also shows that $\text{Suppl}_{\mathbb{Z}\text{-Mod}} \neq \text{Compl}_{\mathbb{Z}\text{-Mod}}$.

Firstly, $\text{E}$ is in $\text{Compl}_{\mathbb{Z}\text{-Mod}}$ by Corollary 3.11, as

$$\text{u.dim}((\mathbb{Z}/p\mathbb{Z}) \oplus \mathbb{Z}) = 2 = 1 + 1 = \text{u.dim}(\mathbb{Z}) + \text{u.dim}(\mathbb{Z}/p^2\mathbb{Z}).$$

It is not in $\text{Suppl}_{\mathbb{Z}\text{-Mod}}$ because if it were, then it would split by Proposition 3.5 as $\text{Rad}(\mathbb{Z}) = 0$. But we cannot have this since it would imply $(\mathbb{Z}/p\mathbb{Z}) \oplus \mathbb{Z} \cong \mathbb{Z} \oplus (\mathbb{Z}/p^2\mathbb{Z})$ which cannot hold by the uniqueness of the cyclic factors of infinite or prime power order in a finitely generated abelian group up to isomorphism.

In fact, this example is obtained through the following considerations which make it clear. Denote $\text{Compl}_{\mathbb{Z}\text{-Mod}}$ shortly by $\mathcal{C}$ and $\text{Suppl}_{\mathbb{Z}\text{-Mod}}$ shortly by $\mathcal{S}$. $\text{Ext}(\mathbb{Z}/p^2\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}/p^2\mathbb{Z}$ (see for example Section 52 in [7] for details on elementary properties of $\text{Ext}_{\mathbb{Z}}$). By Theorem 3.8,

$$\text{Ext}_{\mathcal{C}}(\mathbb{Z}/p^2\mathbb{Z}, \mathbb{Z}) = \text{Rad}(\text{Ext}(\mathbb{Z}/p^2\mathbb{Z}, \mathbb{Z})) \cong \text{Rad}(\mathbb{Z}/p^2\mathbb{Z}) = p\mathbb{Z}/p^2\mathbb{Z} \neq 0.$$ 

But $\text{Ext}_{\mathcal{S}}(\mathbb{Z}/p^2\mathbb{Z}, \mathbb{Z}) = 0$ because any short exact sequence

$$
\begin{array}{c}
0 \rightarrow \mathbb{Z} \rightarrow B \rightarrow \mathbb{Z}/p^2\mathbb{Z} \rightarrow 0
\end{array}
$$

in $\mathcal{S}$ splits by Proposition 3.5 since $\text{Rad} \mathbb{Z} = 0$. So a short exact sequence

$$
\begin{array}{c}
E \in \text{Ext}_{\mathcal{C}}(\mathbb{Z}/p^2\mathbb{Z}, \mathbb{Z}) \cong p\mathbb{Z}/p^2\mathbb{Z} \neq 0
\end{array}
$$
which is not splitting (i.e. the equivalence class of which is a nonzero element in Ext\(_C(\mathbb{Z}/p^2\mathbb{Z}, \mathbb{Z})\)) will give the example we look for. In fact, in the isomorphism Ext(\(\mathbb{Z}/p^2\mathbb{Z}, \mathbb{Z}\)) \(\cong \mathbb{Z}/p^2\mathbb{Z}\), a generator for the cyclic group Ext(\(\mathbb{Z}/p^2\mathbb{Z}, \mathbb{Z}\)) of order \(p^2\) can be taken as the equivalence class \([F]\) of the following short exact sequence

\[
\begin{array}{c}
F : \\
\begin{array}{c}
\mathbb{Z} \rightarrow \mathbb{Z}/p^2\mathbb{Z} \rightarrow 0
\end{array}
\end{array}
\]

where \(u\) is multiplication by \(p^2\) and \(v\) is the canonical epimorphism. The example \(E\) above is in the equivalence class \(p[F]\) whose representative can be taken as the short exact sequence denoted by \(pF\) which is obtained by pushout:

\[
\begin{array}{c}
F : \\
\begin{array}{c}
\mathbb{Z} \rightarrow \mathbb{Z}/p^2\mathbb{Z} \rightarrow 0
\end{array}
\end{array}
\]

\[
\begin{array}{c}
E = pF : \\
\begin{array}{c}
0 \rightarrow \mathbb{Z} \rightarrow (\mathbb{Z}/p\mathbb{Z}) \oplus \mathbb{Z} \rightarrow \mathbb{Z}/p^2\mathbb{Z} \rightarrow 0
\end{array}
\end{array}
\]

where \(h : \mathbb{Z} \rightarrow \mathbb{Z}\) is multiplication by \(p\), i.e. \(h(k) = pk, k \in \mathbb{Z}\), and \(f, g\) are found to be given as in the beginning of the example by pushout computations.

5 The inductive closure of the proper class

\(\text{Suppl}_{\mathbb{Z}\text{-Mod}}\) is \(\text{Compl}_{\mathbb{Z}\text{-Mod}} = \mathcal{N}\text{eat}_{\mathbb{Z}\text{-Mod}}\)

We shall use the following two lemmas in proving this claim; these lemmas say that each \(\mathbb{Z}/p^n\mathbb{Z}\), \(p\) prime and integer \(n \geq 2\), and the Prüfer group \(\mathbb{Z}_{p^\infty}\) (see §3 in [7] for the infinite cocyclic group \(\mathbb{Z}_{p^\infty}\)) are not flat with respect to the proper class \(\text{Suppl}_{\mathbb{Z}\text{-Mod}}\), i.e. not in \(\tau(\text{Suppl}_{\mathbb{Z}\text{-Mod}})\).

**Lemma 5.1.** Let \(p\) be a prime number and \(n \geq 2\) an integer. \(\mathbb{Z}/p^n\mathbb{Z}\) is not flat with respect to following short exact sequence in \(\text{Suppl}_{\mathbb{Z}\text{-Mod}}\):

\[
\begin{array}{c}
E : \\
\begin{array}{c}
0 \rightarrow \mathbb{Z}/p^n\mathbb{Z} \rightarrow (\mathbb{Z}/p\mathbb{Z}) \oplus (\mathbb{Z}/p^{2n-1}\mathbb{Z}) \rightarrow \mathbb{Z}/p^n\mathbb{Z} \rightarrow 0
\end{array}
\end{array}
\]

where

\[
f(k + p^n\mathbb{Z}) = (k + p\mathbb{Z}, kp^{n-1} + p^{2n-1}\mathbb{Z}), \quad k \in \mathbb{Z},
\]

\[
g(a + p\mathbb{Z}, b + p^{2n-1}\mathbb{Z}) = (b - ap^{n-1}) + p^n\mathbb{Z}, \quad a, b \in \mathbb{Z}.
\]

**Proof.** Firstly, \(E\) is in \(\text{Compl}_{\mathbb{Z}\text{-Mod}}\) by Corollary 3.11, as

\[
u.\dim((\mathbb{Z}/p\mathbb{Z}) \oplus (\mathbb{Z}/p^{2n-1}\mathbb{Z})) = 2 = u.\dim(\mathbb{Z}/p^n\mathbb{Z}) + u.\dim(\mathbb{Z}/p^n\mathbb{Z}).
\]

As \(\mathbb{Z}/p^n\mathbb{Z}\) is finite, this sequence \(E\) is in \(\text{Suppl}_{\mathbb{Z}\text{-Mod}}\) by Corollary 4.2. Tensoring \(E\) with \(\mathbb{Z}/p^n\mathbb{Z}\), we do not get a monomorphism \(f \otimes 1_{\mathbb{Z}/p^n\mathbb{Z}}\) (where \(1_{\mathbb{Z}/p^n\mathbb{Z}} : \mathbb{Z}/p^n\mathbb{Z} \rightarrow \mathbb{Z}/p^n\mathbb{Z}\) is the identity map). Because, the element

\[
(1 + p^n\mathbb{Z}) \otimes (p + p^n\mathbb{Z}) \neq 0 \text{ in } (\mathbb{Z}/p^n\mathbb{Z}) \otimes (\mathbb{Z}/p^n\mathbb{Z})
\]

as it corresponds, under the isomorphism \((\mathbb{Z}/p^n\mathbb{Z}) \otimes (\mathbb{Z}/p^n\mathbb{Z}) \cong \mathbb{Z}/p^n\mathbb{Z}\), to the element \(p + p^n\mathbb{Z} \neq 0\) in \(\mathbb{Z}/p^n\mathbb{Z}\) (as \(n \geq 2\)). But

\[
(f \otimes 1_{\mathbb{Z}/p^n\mathbb{Z}})(1 + p^n\mathbb{Z}) \otimes (p + p^n\mathbb{Z}) = (1 + p\mathbb{Z}, p^{n-1} + p^{2n-1}\mathbb{Z}) \otimes (p + p^n\mathbb{Z})
\]

\[
= (1 + p\mathbb{Z}, 0) \otimes (p + p^n\mathbb{Z}) + (0, p^{n-1} + p^{2n-1}\mathbb{Z}) \otimes (p + p^n\mathbb{Z})
\]

\[
= (p + p\mathbb{Z}, 0) \otimes (1 + p^n\mathbb{Z}) + (0, 1 + p^{2n-1}\mathbb{Z}) \otimes (p^n + p^n\mathbb{Z})
\]

\[
= 0 \otimes (1 + p^n\mathbb{Z}) + (0, 1 + p^{2n-1}\mathbb{Z}) \otimes 0 = 0 + 0 = 0
\]
Let \( \text{Im}(\text{Denote by}) \) \( \bigoplus_{p \text{ prime}} (X/pX) \) be a \( \mathbb{N} \)-injective. Motivated by such considerations the following example is found:

**Lemma 5.2.** Let \( p \) be a prime. Denote by \( \mathbb{Q} \) all rational numbers and by \( \mathbb{Q}_p \) the additive subgroup of \( \mathbb{Q} \) consisting all rational numbers whose denominators are relatively prime to \( p \). Then the monomorphism

\[
f : \mathbb{Q}_p \longrightarrow \mathbb{Q} \oplus (\mathbb{Q}_p/p\mathbb{Q}_p)
f(x) = (x, x + p\mathbb{Q}_p), \quad x \in \mathbb{Q}_p
\]

is a \( \text{Suppl}_{\mathbb{Z} \text{-Mod}} \)-monomorphism and \( \mathbb{Z}_p \oplus 0 \) is not flat with respect to this monomorphism.

**Proof.** Denote by \( \mathbb{Q}^{(p)} \), the subgroup of \( \mathbb{Q} \) consisting of all rational numbers whose denominators are powers of the prime \( p \). Observe that \( \mathbb{Q}_p + \mathbb{Q}^{(p)} = \mathbb{Q} \) and \( \mathbb{Q}_p \cap \mathbb{Q}^{(p)} = \mathbb{Z} \).

Also note that \( p\mathbb{Q}_p + \mathbb{Z} = \mathbb{Q}_p \): For \( a/b \in \mathbb{Q}_p \), \( a, b \in \mathbb{Z} \), \( b \neq 0 \), \( p \nmid b \), as the greatest common divisor of \( p \) and \( b \) is 1, there exits \( u, v \in \mathbb{Z} \) such that \( pu + bv = 1 \), so \( a/b = p(ua/b) + va \in p\mathbb{Q}_p + \mathbb{Z} \).

Our claim is that \( \text{Im}(f) \) is a supplement of \( \mathbb{Q}^{(p)} \oplus 0 \) in \( \mathbb{Q} \oplus (\mathbb{Q}_p/p\mathbb{Q}_p) \).

1. \( \text{Im}(f) + (\mathbb{Q}^{(p)} \oplus 0) = \mathbb{Q} \oplus (\mathbb{Q}_p/p\mathbb{Q}_p) \).

   Really for an element \( (y, z + p\mathbb{Q}_p) \in \mathbb{Q} \oplus (\mathbb{Q}_p/p\mathbb{Q}_p) \), where \( y \in \mathbb{Q} \), \( z \in \mathbb{Q}_p \), as \( \mathbb{Q}_p + \mathbb{Q}^{(p)} = \mathbb{Q} \), \( y = u + v \) for some \( u \in \mathbb{Q}_p \) and \( v \in \mathbb{Q}^{(p)} \). Since \( \mathbb{Q}_p = p\mathbb{Q}_p + \mathbb{Z} \), \( u = pu' + k \) and \( z = p'z + n \) for some \( u', z' \in \mathbb{Q}_p \) and \( k, n \in \mathbb{Z} \). So, \( z + p\mathbb{Q}_p = n + p\mathbb{Q}_p = (pu' + n) + p\mathbb{Q}_p \), hence,

\[
(y, z + p\mathbb{Q}_p) = \left(pu' + n, (pu' + n) + p\mathbb{Q}_p + (v + k - n, 0 + p\mathbb{Q}_p)\right)_{f(pu' + n)}
\]

is in \( \text{Im}(f) + (\mathbb{Q}^{(p)} \oplus 0) \) as \( v + k - n \in \mathbb{Q}^{(p)} \) since \( \mathbb{Z} \subseteq \mathbb{Q}^{(p)} \) and \( v \in \mathbb{Q}^{(p)} \).

2. \( \text{Im}(f) \cap (\mathbb{Q}^{(p)} \oplus 0) = [(p\mathbb{Q}_p) \cap \mathbb{Q}^{(p)}] \oplus 0 = (p\mathbb{Z}) \oplus 0 = \ll \text{Im}(f) \)

where the smallness in the end is obtained as follows:

\[
\text{Rad}(\mathbb{Q}_p) = \bigcap_{q \text{ prime}} q\mathbb{Q}_p = p\mathbb{Q}_p
\]

as \( \mathbb{Q}_p \) is \( q \)-divisible for each prime \( q \neq p \) and \( p\mathbb{Z} \leq p\mathbb{Q}_p \), being a cyclic subgroup of the radical, is small in \( \mathbb{Q}_p \); hence, \( f(p\mathbb{Z}) \ll f(\mathbb{Q}_p) = \text{Im}(f) \).

But \( f(p\mathbb{Z}) = p\mathbb{Z} \oplus 0 \).
To prove that \( Z_{p^\infty} \) is not flat with respect to this \( \text{Suppl}_{\mathbb{Z}, \text{Mod}} \)-monomorphism, we will show that \( Q_p \otimes Z_{p^\infty} \) is not 0, while \((Q \oplus (Q_p/pQ_p)) \otimes Z_{p^\infty} = 0\). For elementary properties of tensor product used below see for example §59-60 in [7].

Note that \( Q_p/pQ_p = (Z + pQ_p)/pQ_p \cong Z/(Z \cap pQ_p) = Z/pZ \). So,
\[
[Q \oplus (Q_p/pQ_p)] \otimes Z_{p^\infty} \cong (Q \otimes Z_{p^\infty}) \oplus [(Q_p/pQ_p) \otimes Z_{p^\infty}] = 0
\]
because \( Q \otimes Z_{p^\infty} = 0 \) as \( Q \) is divisible and \( Z_{p^\infty} \) is torsion, and
\[
(Q_p/pQ_p) \otimes Z_{p^\infty} \cong (Z/p\mathbb{Z}) \otimes Z_{p^\infty} = 0
\]
as \( Z/p\mathbb{Z} \) is torsion and \( Z_{p^\infty} \) is divisible.

To prove that \( Q_p \otimes Z_{p^\infty} \neq 0 \), we will show that the element \( 1 \otimes c_0 \neq 0 \) in \( Q_p \otimes Z_{p^\infty} \), where we consider \( Z_{p^\infty} \) to be generated by a sequence of elements \( c_0, c_1, c_2, \ldots \) such that \( pc_0 = 0 \) and \( pc_{n+1} = c_n \) for each integer \( n \geq 0 \). Suppose for the contrary that \( 1 \otimes c_0 = 0 \) in \( Q_p \otimes Z_{p^\infty} \). Then by properties of tensor product, we know that there exists a finitely generated subgroup \( A \leq Q_p \) and a finitely generated subgroup \( B \leq Z_{p^\infty} \) such that \( 1 \in A \), \( c_0 \in B \) and \( 1 \otimes c_0 = 0 \) in \( A \otimes B \), so also \( 1 \otimes c_0 = 0 \) in \( Q_p \otimes B \). Since \( B \neq 0 \) is finitely generated, \( B = \langle c_n \rangle \) for some integer \( n \geq 0 \), so \( B \cong \mathbb{Z}/p^{n+1}\mathbb{Z} \), hence
\[
Q_p \otimes B \cong Q_p \otimes (\mathbb{Z}/p^{n+1}\mathbb{Z}) \cong Q_p/p^{n+1}Q_p,
\]
and under these natural isomorphisms \( 1 \otimes c_0 = 0 \) in \( Q_p \otimes B \) corresponds to \( p^n + p^{n+1}Q_p \) in \( Q_p/p^{n+1}Q_p \) which is not zero and this gives the required contradiction. \( \square \)

**Theorem 5.3.** The inductive closure of the proper class \( \text{Suppl}_{\mathbb{Z}, \text{Mod}} \) is \( \text{Compl}_{\mathbb{Z}, \text{Mod}} = \text{Neat}_{\mathbb{Z}, \text{Mod}} \).

**Proof.** Denote \( \text{Compl}_{\mathbb{Z}, \text{Mod}} \) shortly by \( \mathcal{C} \) and \( \text{Suppl}_{\mathbb{Z}, \text{Mod}} \) shortly by \( \mathcal{S} \). Let \( \mathcal{S} \) be the inductive closure of \( \mathcal{S} \). By a remarkable theorem for abelian groups ([21], [22], [23], Theorem 8.2 (without proof) in [6]), every inductively closed proper class of abelian groups is flatly generated by some subcollection \( \Omega \) of
\[
\{Z/p^k\mathbb{Z} | p \text{ prime and } k \in \mathbb{Z}^+\} \cup \{Z_{p^\infty} | p \text{ prime}\}
\]
So the inductively closed class \( \mathcal{S} = \tau^{-1}(\Omega) \) for some such collection \( \Omega \). To show that \( \mathcal{S} = \mathcal{C} \), it will suffice to show that in the collection \( \Omega \), we can have only \( Z/p^\infty \), \( p \) prime. But that follows from the previous lemmas as each \( \mathbb{Z}/p^n\mathbb{Z} \), \( p \) prime and integer \( n \geq 2 \), and \( Z_{p^\infty} \) are not in \( \tau(\mathcal{S}) \), so also are not in \( \tau(\mathcal{S}) \). Since \( \mathcal{C} \) is inductively closed (as it is flatly generated by \( \{Z/p^k\mathbb{Z} | p \text{ prime}\} \)), by Theorem 3.1 and \( \mathcal{S} \subseteq \mathcal{C} \) by Theorem 3.6, we have \( \mathcal{S} \subseteq \mathcal{C} \), so, of course, all \( \mathbb{Z}/p^\infty \) for each prime \( p \) is in \( \Omega \) and we get that \( \Omega = \{Z/p^k\mathbb{Z} | p \text{ prime}\} \). Hence \( \mathcal{S} = \tau^{-1}(\Omega) = \mathcal{C} \). \( \square \)

**6 The functor \( \text{Ext}_{\text{Suppl}_{\mathbb{Z}, \text{Mod}}} \) is not factorizable**

For a proper class \( \mathcal{A} \) of \( R \)-modules, let us say that \( \text{Ext}_\mathcal{A} \) is factorizable if it is a composition \( H \circ \text{Ext}_R \) for some functor \( H : \mathcal{A}b \longrightarrow \mathcal{A}b \), that is, for all \( R \)-modules \( A, C \),
\[
\text{Ext}_\mathcal{A}(C, A) = H(\text{Ext}_R(C, A)) : \]
Example 6.1. Denote $\mathcal{P}ure_{\mathbb{Z}-\text{Mod}}$ shortly by $\mathcal{P}$. Let $U: \text{Ab} \rightarrow \text{Ab}$ be the Ulm functor which associates to each abelian group $A$, its Ulm subgroup $U(A) = \bigcap_{n=1}^{\infty} nA$. $\text{Ext}_\mathcal{P}$ is denoted by $\text{Pext}$ in §53 of [7] and by Theorem 5.1 in [18], it is shown that for abelian groups $A, C$,

$$\text{Ext}_\mathcal{P}(C, A) = \text{Pext}(C, A) = \bigcap_{n=1}^{\infty} n \text{Ext}(C, A) = U(\text{Ext}(C; A)).$$

Thus, $\text{Ext}_\mathcal{P}$ is factorizable.

Example 6.2. Denote $\text{Comple}_{\mathbb{Z}-\text{Mod}} = \mathcal{N}eat_{\mathbb{Z}-\text{Mod}}$ shortly by $\mathcal{C}$. Consider the functor $\text{Rad}: \text{Ab} \rightarrow \text{Ab}$ which associates with each abelian group $A$, its Fratini subgroup (which is its radical as $\mathbb{Z}$-module and which equals) $\text{Rad}(A) = \bigcap_{\text{prime } p} pA$. Then by Corollary 3.9, since

$$\text{Ext}_\mathcal{C}(C, A) = \bigcap_{\text{prime } p} p \text{Ext}(C, A) = \text{Rad}(\text{Ext}(C; A)),$$

$\text{Ext}_\mathcal{C}$ is factorizable.

But the proper class $\text{Suppl}_{\mathbb{Z}-\text{Mod}}$ behaves badly in this sense:

Theorem 6.3. $\text{Ext}_{\text{Suppl}_{\mathbb{Z}-\text{Mod}}}$ is not factorizable.

Proof. Denote $\text{Suppl}_{\mathbb{Z}-\text{Mod}}$ shortly by $S$ and $\text{Comple}_{\mathbb{Z}-\text{Mod}}$ shortly by $\mathcal{C}$. Suppose for the contrary that $\text{Ext}_S$ is factorizable, so that there exits a functor $H: \text{Ab} \rightarrow \text{Ab}$ such that for all abelian groups $A, C$,

$$\text{Ext}_S(C, A) = H(\text{Ext}_{\mathbb{Z}}(C, A)).$$

In Example 4.4, we have found that $\text{Ext}_S(\mathbb{Z}/p^2\mathbb{Z}, \mathbb{Z}) = 0$. As $\text{Ext}(\mathbb{Z}/p^2\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}/p^2\mathbb{Z}$, this implies that $H(\mathbb{Z}/p^2\mathbb{Z}) = 0$. But also $\text{Ext}(\mathbb{Z}/p^2\mathbb{Z}, \mathbb{Z}/p^2\mathbb{Z}) \cong \mathbb{Z}/p^2\mathbb{Z}$ and as $\mathbb{Z}/p^2\mathbb{Z}$ is a finite group, by Corollary 4.2,

$$\text{Ext}_S(\mathbb{Z}/p^2\mathbb{Z}, \mathbb{Z}/p^2\mathbb{Z}) = \text{ Ext}_C(\mathbb{Z}/p^2\mathbb{Z}, \mathbb{Z}/p^2\mathbb{Z}) = \text{Rad}(\text{Ext}_{\mathbb{Z}}(\mathbb{Z}/p^2\mathbb{Z}, \mathbb{Z}/p^2\mathbb{Z})) \cong p(\mathbb{Z}/p^2\mathbb{Z}).$$

So in this case $H(\mathbb{Z}/p^2\mathbb{Z})$ must be nonzero. This contradiction ends the proof.

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\end{acks}

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TÜMLEYENLERİN DİREKT LİMİTE GÖRE KAPANIŞI

ÖZET

Modüllerde, tamamlayanan (kapalı alt modüller) ve tümleyenler aracılığıyla tanımlanan iki öz sınıfların ve bunların düzenli ve kodüzenli kısa dizilerle ilişkisini incelemekteyiz. Bir Dedekind tam kökü $W$ için, bir $W$-modülin sonlu sürgülü bir alt modülü tamamlayabilir ancak ve ancak tümleyen ise. $\text{Suppl}_{\mathbb{Z}}\text{-Mod}$ öz sınıfinin direkt limite göre kapanış, tüm basit abel grupları ($\mathbb{Z}/p\mathbb{Z}$, $p$ prime) tarafından düz olarak üretilebilir; bu da $\text{Compl}_{\mathbb{Z}}\text{-Mod} = \text{Neat}_{\mathbb{Z}}\text{-Mod}$ öz sınıfin olduğunu gösterir. $\text{Ext}_{\text{Suppl}_{\mathbb{Z}}\text{-Mod}}$ fonktörü, abel gruplarının hiçbir $H : \text{Ab} \rightarrow \text{Ab}$ fonktörü için, $R_{\text{-Mod}} \times R_{\text{-Mod}} \xrightarrow{\text{Ext}_R} \text{Ab} \xrightarrow{H} \text{Ab}$ şeklinde parçalanamaz.