MOTION RELATIVE TO ROTATING AXES
Use of rotating reference axes greatly facilitates the solution of many problems in kinematics where motion is generated within a system or observed from a system which itself is rotating.
Let’s consider the plane motion of two particles $A$ and $B$ in the fixed $X – Y$ plane. We assume that $A$ and $B$ move independently of one another. We observe the motion of $A$ from a moving reference frame $x – y$ which has its origin attached to $B$ and which rotates with an angular velocity $\omega = \dot{\theta}$; the vector notation will be $\vec{\omega} = \omega \vec{k} = \dot{\theta} \vec{k}$ where the vector is normal to the plane of motion and where its positive sense is in the positive $z$ direction according to the right hand rule.
The absolute position vector of $A$ is

$$\vec{r}_A = \vec{r}_B + \vec{r}_{A/B} = \vec{r}_B + \vec{r} = \vec{r}_B + (x\vec{i} + y\vec{j})$$

where $\vec{i}$ and $\vec{j}$ are unit vectors attached to the $x-y$ frame and $\vec{r} = x\vec{i} + y\vec{j}$ stands for $\vec{r}_{A/B}$, the position vector of $A$ with respect to $B$. 
Time Derivatives of Unit Vectors

Velocity and acceleration equations require the time derivatives of the position equation
\[ \vec{r}_A = \vec{r}_B + \vec{r} = \vec{r}_B + (x\vec{i} + y\vec{j}) \]
with respect to time.

Since now the \( \vec{i} \) and \( \vec{j} \) unit vectors rotate with the \( x-y \) axes, their time derivatives will not be zero.
These derivatives are shown in the figure, which shows the infinitesimal change in each unit vector during time $dt$ as the reference axes rotate through an angle $d\theta = \omega dt$. The differential change in $\vec{i}$ is $d\vec{i}$, and it has the direction of $\vec{j}$ and a magnitude equal to the angle $d\theta$ times the length of the vector $\vec{i}$, which is unity. Thus, $d\vec{i} = d\theta \vec{j}$.
Similarly, the unit vector $\vec{j}$ has an infinitesimal change $d\vec{j}$ which points in the negative $x$ direction.

\[
\begin{align*}
\vec{d}i &= \begin{bmatrix} i \\ j \end{bmatrix} d\theta \vec{j} \\
\vec{d}i &= d\theta \vec{j} \\
\frac{d\vec{i}}{dt} &= \frac{d\theta}{dt} \vec{j}
\end{align*}
\]

\[
\begin{align*}
\vec{d}j &= \begin{bmatrix} j \\ i \end{bmatrix} d\theta (-\vec{i}) \\
\vec{d}j &= -d\theta \vec{i} \\
\frac{d\vec{j}}{dt} &= -\frac{d\theta}{dt} \vec{i}
\end{align*}
\]

\[
\begin{align*}
\vec{i} &= \frac{d\vec{i}}{dt} = \omega \vec{j} \\
\vec{j} &= \frac{d\vec{j}}{dt} = -\omega \vec{i}
\end{align*}
\]
By using the cross product, we can see that

\[ \vec{\omega} \times \vec{i} = \vec{\omega} \times \vec{i} = \vec{0} \]

and

\[ \vec{\omega} \times \vec{j} = \vec{\omega} \times \vec{j} = -\vec{\omega} \]

Thus, the time derivative of a unit vector is equal to the product of that unit vector with the angular velocity.
Relative Velocity

We are now going to take the time derivative of the position vector

\[ \vec{r}_A = \vec{r}_B + \vec{r}_{A/B} = \vec{r}_B + \vec{r} = \vec{r}_B + (x\hat{i} + y\hat{j}) \]

Differentiation of the position vectors yields,

\[ \dot{\vec{r}}_A = \dot{\vec{r}}_B + \frac{d}{dt}\left( x\hat{i} + y\hat{j} \right) = \dot{\vec{r}}_B + (\dot{x}\hat{i} + \dot{y}\hat{j} + \dot{x}\hat{i} + \dot{y}\hat{j}) \]

Here

\[ x\hat{i} = \vec{\omega} \times x\hat{i} \]
\[ y\hat{j} = \vec{\omega} \times y\hat{j} \]

\[ \frac{d\vec{r}}{dt} = \vec{\omega} \times (x\hat{i} + y\hat{j}) + \dot{x}\hat{i} + \dot{y}\hat{j} \]
Also, since the observer in $x$-$y$ measures velocity components $\dot{x}$ and $\dot{y}$, we see that $\dot{x} \vec{i} + \dot{y} \vec{j} = \vec{v}_{rel}$, which is the velocity relative to the $x$-$y$ frame of reference. So,

$$\frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r} + \vec{v}_{rel}$$

Thus, the relative velocity equation becomes,

$$\vec{v}_A = \vec{v}_B + \vec{\omega} \times \vec{r} + \vec{v}_{rel}$$
Comparison of this equation with the one obtained for nonrotating axes \( (\vec{v}_A = \vec{v}_B + \vec{v}_{A/B} ) \) shows that

\[
\vec{v}_{A/B} = \vec{\omega} \times \vec{r} + \vec{v}_{rel} .
\]
The term $\vec{v}_B$ is the absolute velocity of $B$ due to the translation of axes $x$-$y$ measured from the fixed point $O$. If the $x$-$y$ axes are not translating but only rotating this velocity would be zero, $\vec{v}_B = 0$. In order to describe the last two terms, let’s analyse the motion of $A$ with respect to rotating axes $x$-$y$. 

$$\vec{v}_A = \vec{v}_B + \vec{\omega} \times \vec{r} + \vec{v}_{rel}$$
\[ \vec{v}_A = \vec{v}_B + \vec{\omega} \times \vec{r} + \vec{v}_{rel} \]

A moves along the curved slot in the plate representing the rotating \( x \)-\( y \) axes. The velocity of \( A \) measured relative to the plate, \( \vec{v}_{rel} \) will be tangent to the path fixed in the \( x \)-\( y \) plane and its magnitude will be equal to \( \dot{s} \) where \( s \) is measured along the path. Its sense will be in the direction of increasing \( s \). This relative velocity may also be viewed as the velocity \( \vec{v}_{A/P} \) relative to a point \( P \) attached to the plate and coincident with \( A \) at the instant under consideration.
The term $\vec{\omega} \times \vec{r}$ appears due to the rotation of the $x$-$y$ axes. It has a magnitude $r \dot{\theta}$ or $r \omega$ and a direction normal to $\vec{r}$. It is the velocity relative to $B$ of point $P$ as seen from nonrotating axes attached to $B$. It is tangent to a circle having a radius $\vec{r}$ at point $A$ (or point $P$ coincident with $A$). Its sense is determined by the right hand rule.
The following comparison will help establish the equivalence of, and clarify the differences between, the relative velocity equations written for rotating and nonrotating reference axes:
Here,

\[ \vec{v}_A = \vec{v}_B + \vec{v}_{P/B} + \vec{v}_{A/P} \]

\[ \vec{v}_A = \vec{v}_P + \vec{v}_{A/P} \]

\[ \vec{v}_A = \vec{v}_B + \vec{v}_{A/B} \]

\[ \vec{v}_{P/B} \] is the term measured from a nonrotating position – otherwise it would be zero.

\[ \vec{v}_{A/P} = \vec{v}_{rel} \] is the velocity of A measured in the x-y frame.

\[ \vec{v}_P \] is the absolute velocity of P and represents the effect of the moving coordinate system, both translational and rotational.

\[ \vec{v}_{A/B} \] is the same as that developed for nonrotating axes.

It is seen that

\[ \vec{v}_{A/B} = \vec{v}_{P/B} + \vec{v}_{A/P} = \vec{\omega} \times \vec{r} + \vec{v}_{rel} \]
Relative Acceleration

The relative acceleration equation may be obtained by differentiating the relative velocity equation, \( \vec{v}_A = \vec{v}_B + \vec{\omega} \times \vec{r} + \vec{v}_{rel} \)

\[
\ddot{a}_A = \ddot{a}_B + \dot{\vec{\omega}} \times \dot{\vec{r}} + \vec{\omega} \times \ddot{\vec{r}} + \frac{d}{dt}(\vec{v}_{rel})
\]

\[
\dot{\vec{r}} = \frac{d}{dt}\left(\begin{array}{c} x\hat{i} + y\hat{j} \\ \vec{r} \end{array}\right) = (\dot{x}\hat{i} + \dot{y}\hat{j}) + (x\ddot{i} + y\ddot{j}) = \vec{\omega} \times \dot{\vec{r}} + \vec{v}_{rel}
\]

So that,

\[
\vec{\omega} \times \dot{\vec{r}} = \vec{\omega} \times (\vec{\omega} \times \dot{\vec{r}} + \vec{v}_{rel}) = \vec{\omega} \times (\vec{\omega} \times \dot{\vec{r}}) + \vec{\omega} \times \vec{v}_{rel}
\]

\[
\frac{d}{dt}(\vec{v}_{rel}) = \dot{\vec{v}}_{rel} = \frac{d}{dt}(\dot{x}\hat{i} + \dot{y}\hat{j}) = (\ddot{x}\hat{i} + \ddot{y}\hat{j}) + (x\dddot{i} + y\dddot{j})
\]

\[
= (\vec{\omega} \times \dddot{x}\hat{i} + \vec{\omega} \times \dddot{y}\hat{j}) + (x\dddot{i} + y\dddot{j})
\]

\[
= \vec{\omega} \times (\dddot{x}\hat{i} + \dddot{y}\hat{j}) + (x\dddot{i} + y\dddot{j})
\]

\[
= \vec{\omega} \times \vec{v}_{rel} + \vec{\omega} \times \vec{a}_{rel}
\]
Adding all these terms yields,

\[
\vec{a}_A = \vec{a}_B + \dot{\omega} \times \vec{r} + \dot{\omega} \times (\ddot{\omega} \times \vec{r}) + 2\ddot{\omega} \times \vec{v}_{rel} + \vec{a}_{rel}
\]

This is the general vector expression for the absolute acceleration of a particle \(A\) in terms of its acceleration \(\vec{a}_{rel}\) measured relative to a moving coordinate system which rotates with an angular velocity \(\ddot{\omega}\) and an angular acceleration \(\dot{\omega} = \vec{\alpha}\).

\(\vec{a}_B\) is the absolute acceleration of the origin of the \(x\)-\(y\) axes \(B\) as measured from the origin \(O\). It arises from the translational motion of the \(x\)-\(y\) reference frame or the planar frame which represents it.
\[ \ddot{a}_A = \ddot{a}_B + \dot{\omega} \times \dot{\mathbf{r}} + \ddot{\omega} \times (\dot{\omega} \times \mathbf{r}) + 2\dot{\omega} \times \dot{\mathbf{v}}_{rel} + \ddot{\mathbf{a}}_{rel} \]

The terms \( \dot{\omega} \times \dot{\mathbf{r}} \) and \( \ddot{\omega} \times (\dot{\omega} \times \mathbf{r}) \) shown in the figure represent, respectively, the tangential and normal components of the acceleration \( \ddot{a}_{P/B} \) of the coincident point \( P \) in its circular motion with respect to \( B \).

This motion would be observed from a set of nonrotating axes moving with \( B \).
The magnitude of $\dot{\omega} \times \vec{r}$ is $r \dot{\theta}$ and its direction is tangent to the circle. The magnitude of $\bar{\omega} \times (\bar{\omega} \times \vec{r})$ is $r \omega^2$ and its direction is from $P$ to $B$ along the normal to the circle. Their senses are determined by the right hand rule. These acceleration components are the components measured by the observer located at $B$ but not rotating with the $x$-$y$ axes.

$$\bar{a}_P = \bar{a}_B + \bar{a}_{P/B}$$

$$\bar{a}_{P/Bn} = \bar{\omega} \times (\bar{\omega} \times \vec{r})$$

$$\bar{a}_{P/Bt} = \bar{\alpha} \times \vec{r}$$
The acceleration of $A$ relative to the plate along the path, $\ddot{a}_{rel}$, may be expressed in rectangular, normal and tangential or polar coordinates in the rotating system. Frequently, $n$ and $t$ components are used as depicted in the figure. The tangential component has the magnitude $\ddot{a}_{rel_t} = \dot{v}_{rel} = \ddot{s}$ where $s$ is the distance measured along the path to $A$. The normal component has the magnitude $\ddot{a}_{rel_n} = \frac{v_{rel}^2}{\rho}$. The sense of this vector is always toward the center of curvature.
**Coriolis Acceleration**

The term $2 \vec{\omega} \times \vec{v}_{rel}$ seen in the figure is called the *Coriolis acceleration*. It represents the difference between the acceleration of $A$ relative to $P$ as measured from nonrotating axes and from rotating axes.

The direction is always normal to the vector $\vec{v}_{rel}$ or $\vec{a}_{rel_t}$ and the sense is established by the right hand rule for the cross product.
The Coriolis acceleration appears when a particle or body translates in addition to its rotation relative to a system which itself is rotating. This translation can be rectilinear or curvilinear.
Rotating versus Nonrotating Systems

The following comparison will help to establish the equivalence of, and clarify the differences between, the relative acceleration equations written for rotating and nonrotating reference axes:
The equivalence of $\ddot{a}_A$ and $\ddot{a}_B + \ddot{\omega} \times \dot{r} + \ddot{\omega} \times (\dot{\omega} \times \dot{r}) + 2\ddot{\omega} \times \ddot{v}_{rel} + \ddot{a}_{rel}$, as shown in the second equation has been described. From the third equation where $\ddot{a}_B + \ddot{a}_{P/B}$ has been combined to give $\ddot{a}_P$, it is seen that the relative acceleration term $\ddot{a}_{A/P}$, unlike the corresponding relative velocity term, is not equal to the relative acceleration $\ddot{a}_{rel}$ measured from the rotating x-y frame of reference.
The Coriolis term is, therefore, the difference between the acceleration $\ddot{a}_A/P$ of $A$ relative to $P$ as measured in a nonrotating system and the acceleration $\ddot{a}_{rel}$ of $A$ relative to $P$ as measured in a rotating system.
\[ \vec{a}_A = \vec{a}_B + \left( \vec{\omega} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + 2\vec{\omega} \times \vec{v}_{rel} + \vec{a}_{rel} \right)_{\vec{a}_{p/B}} + \vec{a}_{A/P} \]

\[ = \vec{a}_B + \vec{a}_{p/B} + \vec{a}_{A/P} \]

\[ = \vec{a}_B + \vec{a}_{A/B} \]

From the fourth equation, it is seen that the acceleration \( \vec{a}_{A/B} \) of \( A \) with respect to \( B \) as measured in a nonrotating system, is a combination of last four terms in the first equation for the rotating system.
Finally the acceleration of $A$ can be expressed by the acceleration of point $P$ coincident with $A$:

$$\vec{a}_A = \vec{a}_P + 2\vec{\omega} \times \vec{v}_{rel} + \vec{a}_{rel}$$

where

$$\vec{a}_P = \vec{a}_B + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$