CENTROIDS AND MOMENTS OF INERTIA
A centroid is a geometrical concept arising from parallel forces. Thus, only parallel forces possess a centroid. Centroid is thought of as the point where the whole weight of a physical body or system of particles is lumped.
If proper geometrical bodies possess an axis of symmetry, the centroid will lie on this axis. If the body possesses two or three symmetry axes, then the centroid will be located at the intersection of these axes.

If one, two or three dimensional bodies are defined as analytical functions, the locations of their centroids can be calculated using integrals.
A composite body is one which is comprised of the combination of several simple bodies. In such bodies, the centroid is calculated as follows:
<table>
<thead>
<tr>
<th>Line – a thin rod (Çizgi – ince çubuk)</th>
<th>Area – a flat plate with constant thickness (Alan – sabit kalınlıklı düz plaka)</th>
<th>Volume – a sphere or a cone (Hacim – küre ya da koni)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Composite</strong></td>
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</tr>
<tr>
<td>$\bar{x} = \int xdl \over \int dl$</td>
<td>$\bar{x} = \sum x_i l_i \over \sum l_i$</td>
<td>$\bar{x} = \int xdv \over \int dv$</td>
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CENTROIDS OF SOME GEOMETRIC SHAPES
Thin Rod in the Shape of a Quarter Circle
Çeyrek Çember Şeklinde İnce Çubuk

\[
\bar{x} = \frac{\int xdl}{\int dl} \quad \bar{y} = \frac{\int ydl}{\int dl}
\]

\[dl = r \cdot d\theta\]

\[
\bar{x}_{el} = x = r \cos \theta \\
\bar{y}_{el} = y = r \sin \theta
\]

\[
\bar{x} = \frac{\int xdl}{\int dl} = \frac{\int_0^{\pi/2} r \cos \theta rd\theta}{\int_0^{\pi/2} rd\theta} = \frac{r^2 [\sin \theta]_0^{\pi/2}}{r[\theta]_0^{\pi/2}} \Rightarrow \bar{x} = \frac{2r}{\pi}
\]

\[
\bar{y} = \frac{\int ydl}{\int dl} = \frac{\int_0^{\pi/2} r \sin \theta rd\theta}{\int_0^{\pi/2} rd\theta} = \frac{r^2 [-\cos \theta]_0^{\pi/2}}{r[\theta]_0^{\pi/2}} \Rightarrow \bar{y} = \frac{2r}{\pi}
\]
**Circular Arc**

![Circular Arc Diagram]

\[ r = \text{radius} \]

\[ \bar{x} = \frac{r \sin \alpha}{\alpha} \]

\[ \bar{y} = 0 \]

**Solid Quarter Circle**

![Solid Quarter Circle Diagram]

\[ \bar{y} = \frac{\int ydA}{\int dA} \]

\[ A = \frac{\pi R^2}{4} \]

\[ dA = \rho d\theta d\rho \]

\[ \int ydA = \int \rho \sin \theta \rho d\rho d\theta = \int_0^{\pi/2} \int_0^R \rho^2 \sin \theta d\rho d\theta = \frac{\rho^3}{3} \bigg|_0^R \int_0^{\pi/2} \sin \theta d\theta \]

\[ = \frac{\rho^3}{3} \bigg|_0^R (-\cos \theta) \bigg|_{\pi/2}^0 = -\frac{R^3}{3} (0-1) = \frac{R^3}{3} \]

\[ x = \frac{R^3}{3} \cdot \frac{4}{\pi R^2} = \frac{4R}{3\pi} \]

\[ \bar{x} = \bar{y} = \frac{4R}{3\pi} \]
Solid Half Circle
İçi Dolu Yarım Daire

\[
\bar{x} = \frac{\int x \, dA}{\int dA} \quad \bar{y} = 0
\]

\[
A = \frac{\pi R^2}{2} \quad dA = \rho d\theta d\rho
\]

\[
\int x \, dA = \int \rho \cos \theta \rho d\rho d\theta = \int_{-\pi/2}^{\pi/2} \int_{0}^{R} \rho^2 \cos \theta \rho d\rho d\theta = \frac{\rho^3}{3} \left[ \int_{0}^{\pi/2} \cos \theta d\theta \right]
\]

\[
\bar{x} = \frac{4R}{3\pi}
\]

Triangle Üçgen

\[
\bar{x} = ? \quad \bar{y} = \frac{h}{3}
\]

dA = w \cdot dy

\[
\frac{b}{h} = \frac{w}{h-y} \quad \Rightarrow \quad w = \frac{b}{h} (h-y) \quad \int dA = \int w \, dy = \int_{0}^{h} \left[ \frac{b}{h} (h-y) \right] dy = \frac{bh}{2}
\]

\[
x = x' + \frac{w}{2} \quad \frac{x'}{y} = \frac{a}{h} \quad \int x \, dA = \int_{0}^{h} \left[ \left( \frac{a}{h} y + \frac{b}{2h} (h-y) \right) \left( \frac{b}{h} (h-y) \right) \right] dy
\]

\[
\bar{x} = \frac{a + b}{3}
\]
Parabola

\[ y = \frac{x^2}{2b} \]

\[ \bar{x} = \frac{\int x \, dA}{\int dA} \]

\[ \bar{y} = \frac{\int y \, dA}{\int dA} \]

\[ \int dA = A = \int y \, dx = \int_{0}^{b} \frac{x^2}{2b} \, dx = \frac{b^2}{6} \]

\[ \int x \, dA = \int x \, y \, dx = \int_{0}^{b} x \frac{x^2}{2b} \, dx = \frac{b^3}{8} \]

\[ \int y \, dA = \int y \, w \, dy = \int_{0}^{b/2} y (b - x) \, dy = \int_{0}^{b/2} y (b - (y^{1/2} 2^{1/2} b^{1/2})) \, dy = \frac{b^3}{40} \]

\[ \bar{x} = \frac{3b}{4} \]

\[ \bar{y} = \frac{3b}{20} \]
Area Between a Line and a Curve

\[ \bar{x} = \frac{\int x \, dA}{\int dA} \]
\[ A = \frac{2a^2}{3} - \frac{a^2}{2} = \frac{a^2}{6} \]
\[ \int x \, dA = \int_0^a x(a^{1/2} x^{1/2} - x) \, dx = \frac{a^{1/2}}{5} \cdot \frac{2}{3} \left[ a^3 - \frac{x^3}{3} \right]_0^a = \frac{2a^3}{5} - \frac{a^3}{3} = \frac{a^3}{15} \quad , \quad \bar{x} = \frac{a^3}{15} \cdot \frac{6}{a^2} = \frac{2a}{5} \]

\[ y = x \quad , \quad y^2 = x^2 \]

\[ y^2 = ax \quad , \quad x^2 = ax \quad , \quad x = a \quad \Rightarrow \quad y = a^{1/2} x^{1/2} \quad , \quad x = \frac{y^2}{a} \]

\[ \bar{y} = \frac{\int y \, dA}{\int dA} \]
\[ \int y \, dA = \int_0^a y \left( x_{right} - x_{left} \right) \, dy = \int_0^a y \left( y - \frac{y^2}{a} \right) \, dy = \frac{y^3}{3} \left[ a - \frac{y^4}{4} \right]_0^a = \frac{a^3}{12} \cdot \frac{6}{a^2} = \frac{a}{2} \]
It is often necessary to calculate the moments of uniformly distributed loads about an axis lying within the plane they are applied to or perpendicular to this plane. Generally, the magnitudes of these forces per unit area (named as pressure or stress) are directly proportional with the distance of their lines of action from the moment axis. This way, an elementary force acting in an elementary area will be proportional to,

\[
\text{distance} \times \text{differential area}
\]
Thus, the total moment:

\[ \int dM = \int d^2 dA, \quad M = \int d^2 dA \]

This integral is named as “area moment of inertia” or “second moment of area”.

Elementary moment is proportional to

distance\(^2\) x differential area: \(dM = d^2 dA\)

\[ dP \approx d, \quad \frac{dF}{dA} \approx d, \quad dF \approx d \cdot dA \]
Moment of inertia is not a physical quantity such as velocity, acceleration or force, but it enables ease of calculation; it is a function of the geometry of the area. Since in Dynamics there is no such concept as the inertia of an area, the moment of inertia has no physical meaning. But in mechanics, moment of inertia is used in the calculation of bending of a bar, torsion of a shaft and determination of the stresses in any cross section of a machine element or an engineering structure.
Rectangular (Cartesian) and Polar Area Moments of Inertia and Product of Inertia
(Kartezyen ve Kutupsal Alan Atalet Momentleri ve Çarpım Alan Atalet Momenti)

By definition, the moments of inertia of area \( dA \) with respect to \( x \) and \( y \) axes are

\[
dI_x = y^2 \, dA \\
dI_y = x^2 \, dA
\]

The moments of inertia of the total area \( A \) with respect to \( x \) and \( y \) axes are

\[
I_x = \int y^2 \, dA \quad \text{The moment of inertia of area } A \text{ with respect to } x \text{ axis} \\
I_y = \int x^2 \, dA \quad \text{The moment of inertia of area } A \text{ with respect to } y \text{ axis}
\]

These moments of inertia are named as “Rectangular (Cartesian) moments of inertia”.
The moment of inertia of area \( dA \) with respect to \( z \) axis or pole \( O \) is by definition

\[
dI_z \left( \text{or } dI_O \text{ or } J \right) = r^2 dA
\]

The moment of inertia of total area \( A \) with respect to \( z \) axis or pole \( O \) is

\[
I_z = \int r^2 dA \quad \text{The moment of inertia of area } A \text{ with respect to } z \text{ axis}
\]

Since the \( z \) axis is perpendicular to the plane of the area and cuts the plane at pole \( O \), the moment of inertia is named “polar moment of inertia”.

\[
r^2 = x^2 + y^2 \quad \text{Therefore,} \quad I_z = I_x + I_y
\]
In certain problems involving unsymmetrical cross sections and in the calculation of moments of inertia about rotated axes, an expression

\[ dI_{xy} = xydA \]

occurs, which has the integrated form

\[ I_{xy} = \int xydA \]

where \( x \) and \( y \) are the coordinates of the element of area \( dA \).

\( I_{xy} \) is named as the "product of inertia" of the area \( A \) with respect to the \( xy \) axes.
Properties of moments of inertia:

1) Area moments of inertia $I_x$, $I_y$, $I_z$ are always positive (+).

2) $I_{xy}$ can be positive (+), negative (-) or zero whenever either of the reference axes is an axis of symmetry, such as the $x$ axis in the figure.

3) The unit for all area moments of inertia is the 4. power of that taken for length $(L^4)$.
Properties of Moments of Inertia:

4) The smallest value of an area moment of inertia that an area can have is realized with respect to an axis that passes from the centroid of this area. The area moment of inertia of an area increases as the area goes further from this axis.

The area moment of inertia will get smaller when the distribution of an area gets closer to the axis as possible.

\[
\begin{align*}
I_x &= 103.13 \text{ } L^4 \\
I_x &= 108 \text{ } L^4 \\
I_x &= 243 \text{ } L^4 \\
I_x &= 752 \text{ } L^4
\end{align*}
\]
Radius of Gyration
Jırasyon (Atalet - Eylemsizlik) Yarıçapı

Consider an area $A$, which has rectangular moments of inertia $I_x$ and $I_y$ and a polar moment of inertia $I_z$ about $O$. We now visualize this area as concentrated into a long narrow strip of area $A$ a distance $k_x$ from the $x$ axis. By definition, the moment of inertia of the strip about the $x$ axis will be the same as that of the original area if

$$k_x^2 A = I_x$$

The distance $k_x$ is called the "radius of gyration" of the area about the $x$ axis.
A similar relation for the y axis is written by considering the area as concentrated into a narrow strip parallel to the y axis as seen in the figure. Also, if we visualize the area as concentrated into a narrow ring of radius $k_z$, we may express the polar moment of inertia as $k_z^2 A = I_z$.

In summary,

\[ I_x = k_x^2 A \quad I_y = k_y^2 A \quad I_z = k_z^2 A \]

\[ k_x = \sqrt{\frac{I_x}{A}} \quad k_y = \sqrt{\frac{I_y}{A}} \quad k_z = \sqrt{\frac{I_z}{A}} \]

Also since,

\[ I_x + I_y = I_z \quad k_x^2 + k_y^2 = k_z^2 \]
Transfer of Axes
Paralel Eksenler (Steiner) Teoremi

The moment of inertia of an area about a noncentroidal axis may be easily expressed in terms of the moment of inertia about a parallel centroidal axis.

In the figure, $\overline{x} - \overline{y}$ axes pass through the centroid $G$ of the area. Let us now determine the moments of inertia of the area about the parallel $xy$ axes. By definition, the moment of inertia of the element $dA$ about the $x$ axis is

$$dI_x = (y_o + d)^2 dA$$

Expanding to the whole area

$$I_x = \int y_o^2 dA + 2d \int y_o dA + d^2 \int dA$$
We see that the first integral is by definition, the moment of inertia $I_x$ about the centroidal $x$ axis. The second integral is zero, since $\int y_o dA = A\bar{y}_o$ and $\bar{y}_o$ is automatically zero with the centroid on the $x$ axis. The third term is simply $Ad^2$. Thus, the expression for $I_x$ and the similar expression for $I_y$ become

$$I_x = \bar{I}_x + Ad^2$$
$$I_y = \bar{I}_y + Ae^2$$

The sum of these two equations give

$$I_z = \bar{I}_z + Ar^2$$
and since $I_x + I_y = I_z$, $\bar{I}_x + \bar{I}_y = \bar{I}_z$
For product of inertia

\[ I_{xy} = \bar{I}_{xy} + Ade \]

The parallel axes theorems also hold for radii of gyration as:

\[ k_z^2 = \bar{k}_z^2 + r^2 \]

where \( \bar{k} \) is the radius of gyration about a centroidal axis parallel to the axis about which \( k \) applies and \( r \) is the perpendicular distance between the two axes. For product of inertia:

\[ k_{xy}^2 = \bar{k}_{xy}^2 + de \]
Two points that should be noted in particular about the transfer of axes are:

- The two transfer axes must be parallel to each other
- One of the axes must pass through the centroid of the area

If a transfer is desired between two parallel axes neither of which passes through the centroid, it is first necessary to transfer from one axis to the parallel centroidal axis and then to transfer from the centroidal axis to the second axis. So, the parallel axis theorem must be used twice.
Rotation of Axes, Principle Axes of Inertia, Principle Moments of Inertia

Eksenlerin Döndürülmesi, Asal Atalet Eksenleri, Asal Atalet Momentleri

In Mechanics it is often necessary to calculate the moments of inertia about rotated axes.

The product of inertia is useful when we need to calculate the moment of inertia of an area about inclined axes. This consideration leads directly to the important problem of determining the axes about which the moment of inertia is a maximum and a minimum.
From the figure, the moments of inertia of the area about the $x'$ and $y'$ axes are:

\[
I'_{x} = \int y^2 \, dA = \int (y \cos \theta - x \sin \theta)^2 \, dA
\]

\[
I'_{y} = \int x^2 \, dA = \int (y \sin \theta + x \cos \theta)^2 \, dA
\]

Expanding and substituting the trigonometric identities:

\[
\cos 2\theta = 1 - 2 \sin^2 \theta
\]

\[
\cos 2\theta = 2 \cos^2 \theta - 1
\]

\[
\sin^2 \theta = \frac{1 - \cos 2\theta}{2}
\]

\[
\cos^2 \theta = \frac{1 + \cos 2\theta}{2}
\]
Defining the relations in terms of $I_x, I_y, I_{xy}$ give

\[
I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\theta - I_{xy} \sin 2\theta \\
I_{y'} = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2} \cos 2\theta + I_{xy} \sin 2\theta
\]

In a similar manner we write the product of inertia about the inclines axes as:

\[
I_{x'y'} = \int x'y'dA = \int (y \cos \theta - x \sin \theta)(y \sin \theta + x \cos \theta)dA
\]

Expanding and substituting the trigonometric identities:

\[
\cos 2\theta = \cos^2 \theta - \sin^2 \theta \quad \text{and} \quad \sin 2\theta = 2 \sin \theta \cos \theta \quad , \quad \sin \theta \cos \theta = \frac{1}{2} \sin 2\theta
\]
Defining the relations for $I_x, I_y, I_{xy}$ give

$$I_{x'y'} = \frac{I_x - I_y}{2} \sin 2\theta + I_{xy} \cos 2\theta$$

Adding, gives

$$I_x + I_y = I_{x'} + I_{y'} = I_z$$

The angle which makes $I_{x'}$ and $I_{y'}$ either maximum or minimum may be determined by setting the derivative of either $I_{x'}$ or $I_{y'}$ with respect to $\theta$ equal to zero. Thus,

$$\frac{dI_{x'}}{d\theta} = -2 \sin 2\theta \frac{I_x - I_y}{2} - 2I_{xy} \cos 2\theta = 0$$
Denoting this critical angle by $\alpha$ gives

$$\tan 2\alpha = \frac{2I_{xy}}{I_y - I_x}$$

The equation gives two values for $2\alpha$ which differ by $\pi$, since $\tan 2\alpha = \tan (2\alpha + \pi)$. Consequently, the two solutions for $\alpha$ will differ by $\pi/2$.

One value defines the axis of maximum moment of inertia and the other value defines the axis of minimum moment of inertia. These two rectangular axes are called “principle axes of inertia”.

When the critical value of $2\theta$ is substituted into the equations, it is seen that the product of inertia is zero for the principle axes of inertia. Substitution of $\sin^2\alpha$ and $\cos^2\alpha$ into the equations, gives the expressions for the principle moments of inertia as:

$$I_{\max_{\min}} = \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}$$
Area Moments of Inertia of Basic Geometric Shapes
1) Rectangle

\[ I_x = \int y^2 \, dA = \int y^2 bdy = b \int_0^h y^2 \, dy = \frac{by^3}{3}\bigg|_0^h = \frac{bh^3}{3} \]

\[ I_y = \int x^2 \, dA = \int x^2 hdx = h \int_0^b x^2 \, dx = \frac{hb^3}{3}\bigg|_0^h = \frac{hb^3}{3} \]

\[ I_x = \bar{I}_x + Ad^2 \quad \text{d} = \frac{h}{2} \quad \bar{I}_x = I_x - Ad^2 = \frac{bh^3}{3} - bh \left( \frac{h}{2} \right)^2 = \frac{bh^3}{12} \]

\[ I_y = \bar{I}_y + Ae^2 \quad \text{e} = \frac{b}{2} \quad \bar{I}_y = I_y - Ae^2 = \frac{hb^3}{3} - bh \left( \frac{b}{2} \right)^2 = \frac{hb^3}{12} \]

\[ I_x = \frac{bh^3}{3} \quad I_y = \frac{hb^3}{3} \quad \bar{I}_x = \frac{bh^3}{12} \quad \bar{I}_y = \frac{hb^3}{12} \]
Area Moments of Inertia of Basic Geometric Shapes

2) Square

Since \( b = h = a \)

\[
I_x = \frac{a^4}{12} \\
I_y = \frac{a^4}{12} \\
\bar{I}_x = \frac{a^4}{12} \\
\bar{I}_y = \frac{a^4}{12}
\]
3) Triangle

\[ I_x = \int y^2 \, dA \quad dA = n \cdot dy \]

From similar triangles \( \frac{n}{h-y} = \frac{b}{h} \) \( \Rightarrow \) \( n = \frac{b}{h} (h-y) \)

\[ I_x = \int_0^h y^2 n \, dy = \int_0^h b \frac{y}{h} (h-y) \, dy = \frac{by^3}{3} \bigg|_0^h - \frac{y^4}{4} \bigg|_0^h \]

\[ I_x = \frac{bh^3}{12} \]

\( \bar{I}_x = I_x - Ad^2 = \frac{bh^3}{12} - \frac{bh}{2} \left( \frac{h}{3} \right)^2 \]

\[ \bar{I}_x = \frac{bh^3}{36} \]

\[ I_y = \int x^2 \, dA = \frac{hb^3}{12} \]

\[ \bar{I}_y = I_y - A e^2 = \frac{hb^3}{36} \]
4) Circle

\[ I_z = \int r^2 \, dA \]

\[ dA = 2\pi r \, dr \]

\[ I_z = \int r^2 \, 2\pi r \, dr = 2\pi \int_0^R r^3 \, dr \]

\[ I_z = 2\pi \frac{R^4}{4} \bigg|_0^R = \frac{2\pi R^4}{4} \]

\[ \bar{I}_z = \frac{\pi R^4}{2} \]

\[ \bar{I}_o = \bar{I}_z = \bar{I}_x + \bar{I}_y \]

Due to symmetry

\[ \bar{I}_x = \bar{I}_y = \frac{\pi R^4}{4} \]

\[ I_x = \bar{I}_x + Ad^2 = \frac{\pi R^4}{4} + \pi R^2 \left( R^2 \right)^2 = \frac{5\pi R^4}{4} \]

\[ I_x = I_y = \frac{5\pi R^4}{4} \]
5) Semicircle

\[ I_x = \frac{\pi R^4}{4} \quad \Rightarrow \quad I_x = \frac{\pi R^4}{8} \]

\[ \bar{I}_x = I_x - Ad^2 = \frac{\pi R^4}{8} - \frac{\pi R^2}{2} \left( \frac{4R}{3\pi} \right)^2, \quad \bar{I}_x = R^4 \left( \frac{\pi}{8} - \frac{8}{9\pi} \right) \]

\[ I_y = \bar{I}_y = \frac{\pi R^4}{4} \quad \Rightarrow \quad I_y = \bar{I}_y = \frac{\pi R^4}{8} \]

6) Quarter circle

\[ I_x = I_y = \frac{\pi R^4}{4} = \frac{\pi R^4}{16} \]
Applications of Area Moment of Inertia
* Determine the area moments of inertia of the area under the curve with respect to x and y axes.

\[ I_x = \ ? \quad I_y = \ ? \]

\[
I_x = \int y^2 \, dA = \int_0^{b/2} y^2 \left( b - 2^{1/2} b^{1/2} y^{1/2} \right) dy = \left[ \frac{b y^3}{3} - \frac{2^{1/2} b^{1/2} y^{7/2}}{7} \cdot 2 \right]_0^{b/2} = \frac{b^4}{168}
\]

\[
I_y = \int x^2 \, dA = \int \frac{x^2}{2b} \, dx = \frac{1}{2b} \cdot \frac{x^5}{5} \bigg|_{0}^{b} = \frac{b^4}{10}
\]
* Determine the area moments of inertia of the area between a curve and a line with respect to x and y axes.

\[
I_y = \int x^2 \, dA, \quad dA = (a^{1/2} x^{1/2} - x) \, dx
\]

\[
\int x^2 \, dA = \int_0^a x^2 (a^{1/2} x^{1/2} - x) \, dx = \left( \frac{a^{1/2} x^{7/2}}{7} \cdot 2 - \frac{x^4}{4} \right) \bigg|_0^a
\]

\[
I_y = \frac{2a^4}{7} - \frac{a^4}{4} = \frac{a^4}{28}
\]

\[
I_x = \int y^2 \, dA = \int_0^a y^2 \left( y - \frac{y^2}{a} \right) \, dy = \left( \frac{y^4}{4} - \frac{y^5}{5 \cdot a} \right) \bigg|_0^a
\]

\[
I_x = \frac{a^4}{4} - \frac{a^4}{5} = \frac{a^4}{20}
\]
** Determine the product of inertias of the following areas.

\[
I_{xy} = \int \int xydA \quad dA = dx\,dy
\]

\[
I_{xy} = \int_{0}^{b} \int_{0}^{h} xydx\,dy = \frac{x^2}{2}\bigg|_{0}^{b} \frac{y^2}{2}\bigg|_{0}^{h} = \frac{b^2h^2}{4}
\]

\[
I_{xy} = \bar{I}_{xy} + Ade, \quad \bar{I}_{xy} = I_{xy} - Ade = \frac{b^2h^2}{4} - bh\left(\frac{b}{2}\right)\left(\frac{h}{2}\right) = 0
\]

The product of inertia is zero whenever either of the centroidal axes is an axis of symmetry.
* Determine the product of inertia of the triangular area.

Considering only the differential rectangular area

\[ dA = ydx \quad \text{, for this area} \quad d\overline{I}_{xy} = 0 \quad \text{and} \quad dI_{xy} = d\overline{I}_{xy} + dAdx \]

\[ dI_{xy} = ydx(x)\left(\frac{y}{2}\right) = x\frac{y^2}{2}dx \quad \text{writing in terms of } x, \quad \frac{y}{b-x} = \frac{h}{b} \quad \text{and} \quad y = \frac{h(b-x)}{b} \]

For the triangle

\[ \int dI_{xy} = I_{xy} = \int_0^b x \frac{h^2}{2b} \left(b^2 - 2bx + x^2\right)dx = \frac{h^2}{2b^2} \left[ \frac{b^2x^2}{2} - \frac{2bx^3}{3} + \frac{x^4}{4} \right]_0^b = \frac{h^2}{2b^2} \left( \frac{b^4}{2} - \frac{2b^4}{3} + \frac{b^4}{4} \right) \]

\[ = \frac{h^2b^2}{4} - \frac{h^2b^2}{8} + \frac{h^2b^2}{8} = \frac{h^2b^2}{24} \]

\[ \overline{I}_{xy} = I_{xy} - Ad\bar{e} = \frac{h^2b^2}{24} - \frac{bh}{2} \left( \frac{h}{3} \right)\left( \frac{b}{3} \right) = \frac{h^2b^2}{24} - \frac{h^2b^2}{18} = \frac{h^2b^2}{72} \]
Products of inertia for various configurations

\[
I_{xy} = \frac{h^2b^2}{8} \quad \bar{I}_{xy} = \frac{h^2b^2}{72}
\]
Products of inertia for various configurations

\[ I_{xy} = ? \quad \bar{I}_{xy} = ? \]

\[ I_{xy} = \int xydA, \quad dA = \rho d\theta d\rho \]

\[ I_{xy} = \int \int_0^{\pi/2} \rho \cos \theta \rho \sin \theta \rho d\theta d\rho \]

\[ \sin 2\theta = 2 \sin \theta \cos \theta \]

\[ = \frac{\rho^4}{4} \left[ \int_0^{\pi/2} \sin \theta \cos \theta d\theta \right] = \frac{R^4}{4} \cdot \frac{1}{2} \int_0^{\pi/2} \sin 2\theta d\theta \quad \left( \int \sin axdx = -\frac{1}{a} \cos ax \right) \]

\[ I_{xy} = \frac{R^4}{8} \left( -\frac{1}{2} \right) \cos 2\theta \bigg|_0^{\pi/2} = -\frac{R^4}{16} (-1-1) = \frac{R^4}{8} \]

\[ \bar{I}_{xy} = I_{xy} - Ad = \frac{R^4}{8} - \frac{\pi R^2}{4} \left( \frac{4R}{3\pi} \right) \left( \frac{4R}{3\pi} \right) = \frac{R^4}{8} - \frac{4R^4}{9\pi} \]