KINETICS OF PARTICLES

IMPULSE AND MOMENTUM
Integration of the equation of motion with respect to time rather than displacement leads to the equations of impulse and momentum. These equations greatly facilitate the solution of many problems in which the applied forces act during extremely short periods of time, as in impact problems, or over specified intervals of time.
Let’s consider the general curvilinear motion in space of a particle of mass \( m \), where the particle is located by its position vector \( \vec{R} \) measured from a fixed origin \( O \).

The velocity of the particle is \( \vec{V} = \dot{\vec{R}} \) and is tangent to its path. The resultant force \( \Sigma \vec{F} \) of all forces on \( m \) is in the direction of its acceleration \( \vec{a} = \dot{\vec{v}} \).
We may write the basic equation of motion for the particle, as

$$\sum \vec{F} = m\ddot{a} = m\ddot{v}$$

or

$$\sum \vec{F} = m\dot{\vec{v}} = \frac{d}{dt}(m\vec{v}) = \frac{d}{dt}(\vec{G}) = \dot{\vec{G}}$$

Where the product of the mass and velocity is defined as the linear momentum of the particle. This equation states that the resultant of all forces acting on a particle equals its time rate of change of linear momentum.
In SI, the unit of linear momentum $m\vec{v}$ is kg\cdot m/s, which also equals N\cdot s.

Since the equation of impulse and momentum is a vector equation, in addition to the equality of the magnitudes of $\Sigma F$ and $\dot{G}$, the direction of the resultant force coincides with the direction of the rate of change in linear momentum, which is the direction of the rate of change of velocity. Linear impulse momentum equation is one of the most useful and important relationships in dynamics, and it is valid as long as mass $m$ of the particle is not changing with time.
We now write the three scalar components of linear momentum equation as

\[ \sum F_x = \dot{G}_x \quad \sum F_y = \dot{G}_y \quad \sum F_z = \dot{G}_z \]

These equations may be applied independently of one another.
All that we have done so far is to rewrite Newton’s second law in an alternative form in terms of momentum. But we may describe the effect of the resultant force $\sum \mathbf{F}$ on the linear momentum of the particle over a finite period of time simply by integrating the linear momentum equation with respect to time $t$. Multiplying the equation by $dt$ gives

$$\sum \mathbf{F} dt = d\mathbf{G},$$

which we integrate from time $t_1$ to time $t_2$ to obtain

$$\int_{t_1}^{t_2} \sum \mathbf{F} dt = \int (d\mathbf{G}) = \mathbf{G}_2 - \mathbf{G}_1 = \Delta \mathbf{G}.$$
Here the linear momentum at time $t_2$ is $G_2 = mv_2$ and the linear momentum at time $t_1$ is $G_1 = mv_1$. The product of force and time is defined as the linear impulse of the force, and this equation states that the total linear impulse on $m$ equals the corresponding change in linear momentum of $m$.

Alternatively, we may write

$$\vec{G}_1 + \sum \int \vec{F} dt = \vec{G}_2$$
which says that the initial linear momentum of the body plus the linear impulse applied to it equals its final linear momentum.

\[ \vec{G}_1 = m\vec{v}_1 + \int \sum \vec{F} dt = \vec{G}_2 = m\vec{v}_2 \]
The impulse integral is a vector which, in general, may involve changes in both magnitude and direction during the time interval. Under these conditions, it will be necessary to express $\sum \vec{F}$ and $\vec{G}$ in component form and then combine the integrated components. The components become the scalar equations, which are independent of one another.

\[
\begin{align*}
\int_{t_1}^{t_2} F_x dt &= (mv_x)_2 - (mv_x)_1 = G_{x_2} - G_{x_1} = \Delta G_x \\
\int_{t_1}^{t_2} F_y dt &= (mv_y)_2 - (mv_y)_1 = G_{y_2} - G_{y_1} = \Delta G_y \\
\int_{t_1}^{t_2} F_z dt &= (mv_z)_2 - (mv_z)_1 = G_{z_2} - G_{z_1} = \Delta G_z
\end{align*}
\]
In some cases, certain forces are very large and of short duration. Such forces are called \textit{impulsive forces}. An example is a force of sharp impact. We frequently assume that impulsive forces are constant over their time of duration, so that they can be brought outside of the linear impulse momentum integral. In addition, we frequently assume that \textit{nonimpulsive forces} can be neglected in comparison with impulsive forces. An example of a nonimpulsive force is the weight of a baseball during its collision with a bat - the weight of the ball, about 1.425 N, is small compared with the force exerted on the ball by the bat, which is about several thousand Newtons in magnitude.
There are cases where a force acting on a particle changes with the time in a manner determined by experimental measurements or by other approximate means. In this case, a graphical or numerical integration must be performed. If, for example, a force $F$ acting on a particle in a given direction changes with the time $t$ as indicated in the figure, the impulse, of this force from $t_1$ to $t_2$ is the shaded area under the curve.
If the resultant force on a particle is zero during an interval of time, its linear momentum $G$ remains constant. In this case, the linear momentum of the particle is said to be conserved. Linear momentum may be conserved in one direction, such as $x$, but not necessarily in the $y$- or $z$- directions.

$$\Delta \vec{G} = 0 \implies \vec{G}_1 = \vec{G}_2$$

$$m\vec{v}_1 = m\vec{v}_2$$

This equation expresses the principle of conservation of linear momentum.
In addition to the equations of linear impulse and linear momentum, there exists a parallel set of equations for angular impulse and angular momentum. First, we define the term **angular momentum**. The figure shows a particle $P$ of mass $m$ moving along a curve in space.

The particle is located by its position vector $\vec{r}$ with respect to a convenient origin $O$ of fixed coordinates $x$-$y$-$z$. 

$H_O = \vec{r} \times m\vec{v}$
The velocity of the particle is $\vec{v} = \dot{\vec{r}}$, and its linear momentum is $\vec{G} = m\vec{v}$. The moment of the linear momentum vector $m\vec{v}$ about the origin $O$ is defined as the angular momentum $\vec{H}_O$ of $P$ about $O$ and is given by the cross-product relation for the moment of a vector.

$$\vec{H}_O = \vec{r} \times m\vec{v} = \vec{r} \times \vec{G}$$
The angular momentum then is a vector perpendicular to the plane $A$ defined by $\vec{r}$ and $\vec{v}$. The sense of $\vec{H}_O$ is clearly defined by the right-hand rule for cross products.

\[ \vec{H}_O = \vec{r} \times m\vec{v} \]
The scalar components of angular momentum may be obtained from the expansion

\[ \vec{H}_o = \vec{r} \times m\vec{\nu} = m(v_z y - v_y z)i + m(v_x z - v_z x)j + m(v_y x - v_x y)k \]

or

\[ \vec{H}_o = m \begin{vmatrix} i & j & k \\ x & y & z \\ v_x & v_y & v_z \end{vmatrix} \]

so that

\[ H_{ox} = m(v_z y - v_y z) \quad H_{oy} = m(v_x z - v_z x) \quad H_{oz} = m(v_y x - v_x y) \]
Each of these expressions for angular momentum may be checked from the figure, which shows the three linear momentum components, by taking the moments of these components about the respective axes.

In SI units, angular momentum has the units kg·m²/s = N·m·s.
Rate of Change of Angular Momentum

If $\sum \vec{F}$ represents the resultant of all forces acting on the particle $P$, the moment $\vec{M}_o$ about the origin $O$ is the vector cross product

$$\sum \vec{M}_o = \vec{r} \times \sum \vec{F} = \vec{r} \times (m\vec{v})$$

We now differentiate $\vec{H}_o = \vec{r} \times m\vec{v}$ with time, using the rule for the differentiation of a cross product and obtain

$$\dot{\vec{H}}_o = \frac{d}{dt} \left( \vec{r} \times m\vec{v} \right) = \left\{ \begin{array}{c} \dot{\vec{r}} \times m\vec{v} \\ \vec{r} \times \dot{m}\vec{v} \end{array} \right\} = \left\{ \begin{array}{c} m\vec{a} \\ \vec{M}_o \end{array} \right\}$$

The term $\vec{v} \times m\vec{v}$ is zero since the cross product of parallel vectors is zero.
Substitution into the expression for moment about $O$ gives

$$\sum \vec{M}_o = \ddot{H}_o$$

This equation states that the moment about the fixed point $O$ of all forces acting on $m$ equals the time rate of change of angular momentum of $m$ about $O$. This relation, particularly when extended to a system of particles, rigid or nonrigid, provides one of the most powerful tools of analysis in dynamics.

The scalar components of this equation are

$$\sum M_{ox} = \ddot{H}_{ox} \quad \sum M_{oy} = \ddot{H}_{oy} \quad \sum M_{oz} = \ddot{H}_{oz}$$
The Angular Impulse-Momentum Principle

To obtain the effect of the moment on the angular momentum of the particle over a finite period of time, we integrate \( \sum \vec{M}_o = \dot{\vec{H}}_o \) from time \( t_1 \) to \( t_2 \).

\[
\int_{t_1}^{t_2} \sum \vec{M}_o \, dt = \int \dot{\vec{H}}_o = (\vec{H}_o)_2 - (\vec{H}_o)_1 = \Delta \vec{H}_o
\]

or

\[
\int_{t_1}^{t_2} \sum \vec{M}_o \, dt = (\vec{r}_2 \times m\vec{v}_2) - (\vec{r}_1 \times m\vec{v}_1) = \Delta \vec{H}_o
\]

\( t_1 \) \quad \text{total angular impulse}
\[
\text{change in angular momentum}
\]
The product of moment and time is defined as angular impulse and this equation states that the total angular impulse on \( m \) about the fixed point \( O \) equals the corresponding change in angular momentum of \( m \) about \( O \).

Alternatively, we may write

\[
\left( \vec{H}_o \right)_1 + \int_{t_1}^{t_2} \sum \vec{M}_o \, dt = \left( \vec{H}_o \right)_2
\]
Plane-Motion Applications

Most of the applications can be analyzed as plane-motion problems where moments are taken about a single axis normal to the plane motion. In this case, the angular momentum may change magnitude and sense, but the direction of the vector remains unaltered.

\[
\int_{t_1}^{t_2} \sum M_0 \, dt = (H_o)_2 - (H_o)_1
\]

\[
\int \sum Fr \sin \theta \, dt = m v_2 d_2 - m v_1 d_1
\]
Conservation of Angular Momentum

If the resultant moment about a fixed point \( O \) of all forces acting on a particle is zero during an interval of time, its angular momentum \( \vec{H}_O \) remains constant. In this case, the angular momentum of the particle is said to be conserved. Angular momentum may be conserved about one axis but not about another axis.

\[
\Delta \vec{H}_O = 0 \implies \vec{H}_{O_1} = \vec{H}_{O_2}
\]

This equation expresses the principle of conservation of angular momentum.