ROTATIONAL STRESSES
INTRODUCTION

High centrifugal forces are developed in machine components rotating at a high angular speed of the order of 100 to 500 revolutions per second (rps). High centrifugal force produces high hoop and radial stresses in machine components such as rotors and blades of steam and gas turbines. It is almost necessary to analyse these stresses in such components because these are subjected to fatigue loading and creep strain. Many a times a component fails due to crack developed by fatigue and creep during operation. In this part we will analyse hoop and radial stresses in components like thin disc, long cylinder, or a disc of uniform strength rotating at high angular speeds.
ROTATING RING

Consider a ring of mean radius $R$, rotating about its axis $O$, with circular speed $\omega$ as shown in Figure 1. Say the section of the ring is rectangular with breadth $b$ and thickness $t$ as shown in the figure. Take a small element $abcd$, subtending an angle $d\theta$ at the centre, at angular displacement $\theta$ from $x-x$ axis.

Volume of small element $= Rd\theta bt$
if weight density of ring $= \rho$
weight of small element $= \rho Rbtd\theta$

Figure 1 Rotating ring
\(\text{d}F\), centrifugal force acting on this element =

\[ \sum F = ma \quad \Rightarrow \quad \left( \frac{\rho R \omega dt \theta}{g} \right) \omega^2 R \]

where \(\rho\) = weight density,
\(\omega\) = angular velocity in rad/s, and
\(g\) = acceleration due to gravity.

Vertical component of centrifugal force =

\[ \left( \frac{\rho \omega^2 R^2 b t}{g} \right) d\theta \sin \theta \]

Horizontal component of \(\frac{\rho \omega^2 R^2 b t}{g} d\theta \cos \theta \)

Horizontal component of centrifugal force will be cancelled when we consider another small element \(a'b'c'd'\) in second quadrant at an angle \(\theta\), but the vertical component of \(\text{d}F\) will be added.
Total vertical component or the bursting force on ring in vertical direction acting on horizontal diameter \( xx \):

\[
\sigma_c = \frac{\rho \omega^2 R^2 b t}{g} \sin \theta d \theta = \frac{\rho \omega^2 R^2 b t}{g} \left. -\cos \theta \right|_0^\pi = \frac{2 \omega^2 R^2 b t \rho}{g}
\]

Say, \( \sigma_c \) is the hoop stress developed in horizontal section along \( xx \).

Area of cross section resisting the force = \( 2bt \)

Resisting force (Eql.in y direction) = \( 2\sigma_c b t = \text{Acting force} = \frac{2 \omega^2 R^2 b t \rho}{g} \)

Circumferential or hoop stress developed in section of ring,

\[
\sigma_c = \frac{\rho \omega^2 R^2}{g} = \frac{\rho V^2}{g}
\]

where \( V = \) linear velocity of ring at radius \( R = \omega R \).
Example 8.1

To develop substantial stress in ring, \( \omega \) has to be very high.
Let us take a ring of mild steel with weight density of 7644 kg/m\(^3\) or 76.44 \( \times \) \( 10^3 \) N/m\(^3\).
Say the yield strength (yield point) of mild steel is 280 N/mm\(^2\) and let us take
\[ \sigma_{yp} = \sigma_c, \] developed in ring.

\[
\sigma_c = \frac{\rho \omega^2 R^2}{g} = \frac{\rho V^2}{g}
\]

\[ V^2 = \frac{\sigma_c g}{\rho} = \frac{280 \times 10^6 \times 9.81}{76.44 \times 10^3} \text{ m}^2/\text{s}^2 \]

\[ V^2 = 3.593 \times 10^4 \]

\[ V = 1.8956 \times 100 \text{ m/s} = 189.56 \text{ m/s} \]

If we take the ring radius equal to 0.5 m, then angular speed required to develop hoop stress of the order of 280 N/mm\(^2\) is

\[
\omega = \frac{V}{R} = \frac{189.56}{0.5} = 379.13 \text{ rad/sec.}
\]

or

3620 revolutions per minute.
STRESSES IN A THIN ROTATING DISC

Consider a thin disc of inner radius $R_1$ and outer radius $R_2$, rotating at an angular speed $\omega$ about its axis $O$. The thickness $t$ of the disc is small and it is assumed that stresses in the disc along the thickness do not vary and there is no axial stress developed in the disc; in other words the disc is under plane stress conditions. Take a small element $abcd$ at a radius $r$, of thickness $dr$, subtending an angle $d\theta$ at the centre of the disc (Figure 2).

When the disc is rotating at a high speed, say radius $r$ changes to $r + u$ and the radius $r + dr$ changes to $r + u + dr + du$. In other words, change in radius $r$ is $u$ and change in radial thickness $dr$ is $du$. Say circumferential stress developed is $\sigma_c$ at radius $r$.

**Figure 2** Thin rotating disc
Radial stress at radius $r$ is $\sigma_c$.
Say $\rho = \text{weight density of thin disc}$. weight of the small element $= \rho (rd\theta)t \, dr$.
Centripetal force on small element,

$$CF = \frac{\rho rt \, dr \, d\theta}{g} \omega^2 r$$

Circumferential forces on faces $ad$ and $bc = \sigma_c \, dr \, t$.
Radial force on face $ab = \sigma_r r \, d\theta \, t$.
Radial force on face $cd = (\sigma_r + d\sigma_r)(r + dr) \, d\theta \, t$.
Resolving the forces along the vertical direction $OCF$ as shown in Figure (b):

$$\sigma_r r \, d\theta \, t + 2\sigma_c \sin \frac{d\theta}{2} \, dt \, t = (\sigma_r + d\sigma_r)(r + dr) t \, d\theta + \frac{\rho \omega^2 r^2 t \, dr \, d\theta}{g}$$

But angle $d\theta \to 0$ is very small

$$\sin \frac{d\theta}{2} \approx \frac{d\theta}{2}$$
Moreover, $r \, d\theta$ is common on both sides; simplifying and neglecting the term $d\sigma_r \, dr$, we get

$$\sigma_c \, dr = r \, d\sigma_r + \sigma_r \, dr + \frac{\rho \omega^2 r^2 \, dr}{g}$$

Dividing this equation throughout by $dr$, we get

$$\sigma_c = r \frac{d\sigma_r}{dr} + \sigma_r + \frac{\rho \omega^2 r^2}{g}$$

or

$$\sigma_c - \sigma_r = r \frac{d\sigma_r}{dr} + \frac{\rho \omega^2 r^2}{g} \quad \text{******}$$

Now considering the strains in circumferential and radial directions

Circumferential strain, (Hooke’s law)

$$\varepsilon_c = \frac{r + u - r}{r} = \frac{u}{r} = \frac{\sigma_c}{E} - \nu \frac{\sigma_r}{E}$$
where $E$ is Young’s modulus and $v$ is Poisson’s ratio.

Radial strains,

$$
\varepsilon_r = \frac{dr + du - dr}{r} = \frac{du}{dr} = \frac{\sigma_r - v \sigma_c}{E}
$$

or

$$
\frac{r}{E} \left( \sigma_c - v \sigma_r \right) = \frac{du}{dr} = \frac{1}{E} \left( \sigma_r - v \sigma_c \right)
$$

Differentiating $\frac{u}{r} = \frac{1}{E} \left( \sigma_c - v \sigma_r \right)$ with respect to $r$, we get

$$
u = \frac{r}{E} \left( \sigma_c - v \sigma_r \right)
$$

$$
\frac{du}{dr} = \frac{1}{E} \left[ \left( \sigma_c - v \sigma_r \right) + \frac{r}{E} \left( \frac{d\sigma_c}{dr} - \frac{vd\sigma_r}{dr} \right) \right]
$$

Equating these last two equations, we get

$$
\frac{1}{E} \left( \sigma_r - v \sigma_c \right) = \frac{1}{E} \left[ \left( \sigma_c - v \sigma_r \right) + \frac{r}{E} \left( \frac{d\sigma_c}{dr} - \frac{vd\sigma_r}{dr} \right) \right]
$$
After the simplification of this equation

\[(\sigma_c - \sigma_r)(1+v) = r \left[ v \frac{d\sigma_r}{dr} - \frac{d\sigma_c}{dr} \right] \]

Putting the value of \((\sigma_c - \sigma_r)\) from \(\sigma_c - \sigma_r = r \frac{d\sigma_r}{dr} + \frac{\rho \omega^2 r^2}{g}\) into above eq.

\[ \left( r \frac{d\sigma_r}{dr} + \frac{\rho \omega^2 r^2}{g} \right) (1+v) = x \left[ v \frac{d\sigma_r}{dr} - \frac{d\sigma_c}{dr} \right] \]

Simplifying this equation

\[ \frac{d\sigma_r}{dr} + \frac{d\sigma_c}{dr} = -(1+v) \frac{\rho \omega^2 r}{g} \]

or

\[ \frac{d}{dr} (\sigma_r + \sigma_c) = -(1+v) \frac{\rho \omega^2 r}{g} \]

Integrating this we get

\[ \sigma_c + \sigma_r = -(1+v) \frac{\rho \omega^2 r^2}{2g} + A \]

where \(A\) is constant of integration.
From \( \sigma_c - \sigma_r = r \frac{d\sigma_r}{dr} + \frac{\rho \omega^2 r^2}{g} \)

\[
\sigma_c = \sigma_r + r \frac{d\sigma_r}{dr} + \frac{\rho \omega^2 r^2}{g}
\]

Putting this value in \( \sigma_c + \sigma_r = -(1+v) \frac{\rho \omega^2 r^2}{2g} + A \) we get

\[
\sigma_r + r \frac{d\sigma_r}{dr} + \frac{\rho \omega^2 r^2}{g} + \sigma_r = -(1+v) \frac{\rho \omega^2 r^2}{2g} + A
\]
or

\[
2\sigma_r + r \frac{d\sigma_r}{dr} = -(3+v) \frac{\rho \omega^2 r^2}{2g} + A
\]

Multiplying this by \( r \) throughout, we get

\[
2rd\sigma_r + r^2 \frac{d\sigma_r}{dr} = -(3+v) \frac{\rho \omega^2 r^3}{2g} + Ar
\]

Integrating this equation,

\[
r^2\sigma_r = -(3+v) \frac{\rho \omega^2 r^4}{8g} + \frac{Ar^2}{2} + B
\]

where \( B \) is another constant of integration.
or
\[ \sigma_r = -(3 + v) \frac{\rho \omega^2 r^2}{8g} + \frac{A}{2} + \frac{B}{r^2} \]

Radial stress,
\[ \sigma_r = \frac{B}{r^2} + \frac{A}{2} - (3 + v) \frac{\rho \omega^2 r^2}{8g} \]

From \( \sigma_c + \sigma_r = -(1 + v) \frac{\rho \omega^2 r^2}{2g} + A \)

\[ \sigma_c = A - (1 + v) \frac{\rho \omega^2 r^2}{2g} - \sigma_r \]

Putting the value of \( \sigma_r \), we get the hoop stress as
\[ \sigma_c = A - (1 + v) \frac{\rho \omega^2 r^2}{2g} - \frac{B}{r^2} - \frac{A}{2} + (3 + v) \frac{\rho \omega^2 r^2}{8g} \]

\[ = \frac{A}{2} - \frac{B}{r^2} - (1 + 3v) \frac{\rho \omega^2 r^2}{8g} \]

Let us take \( \frac{3 + v}{8} = k_1 \), constant, and \( \frac{1 + 3v}{8} = k_2 \), another constant.
Expressions for stresses will now be:

Radial stress

\[ \sigma_r = \frac{A}{2} + \frac{B}{r^2} - k_1 \frac{\rho \omega^2 r^2}{g} \]

\[ \sigma_c = \frac{A}{2} - \frac{B}{r^2} - k_2 \frac{\rho \omega^2 r^2}{g} \]

Constants \( A \) and \( B \) can be determined by using boundary conditions of a thin hollow disc:

Radial stress \( \sigma_r = 0 \) at \( r = R_1 \); and at \( r = R_2 \)

(normal stress on free surfaces is zero), therefore

\[ 0 = \frac{A}{2} + \frac{B}{R_1^2} - k_1 \frac{\rho \omega^2 R_1^2}{g} \]

\[ 0 = \frac{A}{2} + \frac{B}{R_2^2} - k_1 \frac{\rho \omega^2 R_2^2}{g} \]
From these two equations, values of constants are

\[ B = -k_1 \frac{\rho \omega^2 R_1^2 R_2^2}{g} \]
\[ \frac{A}{2} = +k_1 \frac{\rho \omega^2 r^2}{g} \left( R_1^2 + R_2^2 \right) \]

Final expressions for stresses for hollow disc are

\[
\sigma_r = k_1 \frac{\rho \omega^2}{g} \left( R_1^2 + R_2^2 \right) - k_1 \frac{\rho \omega^2}{g} \times \frac{R_1^2 R_2^2}{r^2} - k_2 \frac{\rho \omega^2 r^2}{g}
\]
\[
\sigma_c = k_1 \frac{\rho \omega^2}{g} \left( R_1^2 + R_2^2 \right) + k_1 \frac{\rho \omega^2 r^2 R_1^2 R_2^2}{r^2} - k_2 \frac{\rho \omega^2 r^2}{g}
\]

where

\[ k_1 = \frac{3 + \nu}{8}; \quad k_2 = \frac{1 + 3\nu}{8} \]

For radial stress to be maximum, \( \frac{d\sigma_r}{dr} = 0 \)

or

\[ +2k_1 \frac{\rho \omega^2}{g} \times \frac{R_1^2 R_2^2}{r^3} - 2k_1 \frac{\rho \omega^2 r}{g} = 0 \]
or

\[ r^4 = R_1^2 R_2^2 \]
\[ r = \sqrt{R_1 R_2}, \quad \text{putting this value of } r \]
\[
\sigma_{r_{\text{max}}} = k_1 \frac{\rho \omega^2}{g} \left( R_1^2 + R_2^2 \right) - k_1 \frac{\rho \omega^2}{g} \times \frac{R_1 R_2 R_1^2}{R_1 R_2} - k_1 \frac{\rho \omega^2}{g} \times R_1 R_2
\]
\[ = \frac{k_1 \rho \omega^2}{g} (R_2 - R_1)^2 \]

From the expression for \( \sigma_c \), it can be seen that the maximum circumferential stress will occur at \( r \) equal to a minimum value, \( \text{i.e. } r = R_1, \) (inner radius).

\[
\sigma_{c_{\text{max}}} = k_1 \rho \omega^2 \left( \frac{R_1^2 + R_2^2}{g} \right) + k_1 \omega^2 r^2 \frac{R_1 R_2^2}{R_1^2} - k_2 \frac{\rho \omega^2 R_1^2}{g}
\]
\[ = \frac{\rho \omega^2}{g} \left[ k_1(2R_2^2 + R_1^2) - k_2 R_1^2 \right] \]
**Solid Disc**

In a solid disc, at \( r = 0 \), i.e. at the centre, stresses cannot be infinite, therefore constant \( B \) has to be zero otherwise \( \frac{B}{r^2} \) will become infinite at the centre.

So, stresses are

\[
\sigma_r = \frac{A}{2} - k_1 \frac{\rho \omega^2 r^2}{g} \\
\sigma_c = \frac{A}{2} - k_2 \frac{\rho \omega^2 r^2}{g}
\]

Say outer radius of solid disc = \( R \).

At outer surface, radial stress \( \sigma_r = 0 \). Therefore,

\[
0 = \frac{A}{2} - k_1 \frac{\rho \omega^2 R^2}{g}
\]

Constant,

\[
\frac{A}{2} = k_1 \frac{\rho \omega^2 R^2}{g}
\]

Stresses at any radius \( r \),

\[
\sigma_r = k_1 \frac{\rho \omega^2 R^2}{g} - k_1 \frac{\rho \omega^2 r^2}{g} = k_1 \rho \omega^2 \left( R^2 - r^2 \right)
\]

\[
\sigma_c = k_1 \frac{\rho \omega^2 R^2}{g} - k_2 \frac{\rho \omega^2 r^2}{g} = \rho \omega^2 k_1 R^2 - k_2 r^2
\]

obviously both the stresses are maximum at the centre, i.e. at \( r = 0 \)

\[
\sigma_{r_{\text{max}}} = \sigma_{c_{\text{max}}} = \frac{k_1 \rho \omega^2 R^2}{g}
\]
Example 8.2

A thin uniform steel disc of diameter 500 mm is rotating about its axis at 3000 rpm. Calculate the maximum principal stress and maximum in plane shear stress in the disc. Draw the circumferential stress and radial stress distribution along the radius of the thin disc.

Given Poisson’s ratio, \( \nu = 0.3 \), \( \rho = 7700 \text{ kg/m}^3 \) \( g = 9.81 \text{ m/s}^2 \)

Radius, \( R = 250 \text{ mm} \)

\[
\rho = 7700 \times 9.81 \times 10^{-9} = 7.456 \times 10^{-5} \text{ N/mm}^3
\]

\[
\omega = \frac{2\pi \times 3000}{60} = 314.16 \text{ rad/s}
\]

Constants

\[
k_1 = \frac{3 + \nu}{8} = \frac{3.3}{8}
\]

\[
k_2 = \frac{1 + 3\nu}{8} = \frac{1.9}{8}
\]

Maximum principal stress \( \sigma_r, \sigma_c \) at centre

\[
\sigma_{r_{\text{max}}} = \sigma_{c_{\text{max}}} = k_1 \frac{\rho \omega^2 R^2}{g} = \frac{3.3}{8} \times \frac{7.456 \times 10^{-5} \times (314.16)^2 \times 250^2}{9810}
\]

\[
= 19.34 \text{ N/mm}^2
\]
Expression for radial stress

\[ \sigma_r = \frac{k_1 \rho \omega^2}{g} [R^2 - r^2] = 3.0943 \times 10^{-4} [R^2 - r^2] \]

<table>
<thead>
<tr>
<th>( \sigma_r ) (MPa)</th>
<th>0</th>
<th>6.962</th>
<th>12.377</th>
<th>16.245</th>
<th>18.566</th>
<th>19.34</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R ) (mm)</td>
<td>250</td>
<td>200</td>
<td>150</td>
<td>100</td>
<td>50</td>
<td>0</td>
</tr>
</tbody>
</table>

Hoop stress

\[ \sigma_c = \frac{k_1 \rho \omega^2 R^2}{g} - k_2 \frac{\rho \omega^2 r^2}{g} = \frac{\rho \omega^2}{g} \left[ k_1 R^2 - k_2 r^2 \right] \\
= \frac{7.456 \times 10^{-5} \times (314.16)^2}{9810} \left[ \frac{3.3}{8} \times 250^2 - \frac{1.9}{8} \times r^2 \right] \\
= 7.50 \times 10^{-4} \left[ 25781.25 - 0.2375 r^2 \right] \]

<table>
<thead>
<tr>
<th>( \sigma_c ) (MPa)</th>
<th>8.20</th>
<th>12.21</th>
<th>15.33</th>
<th>17.55</th>
<th>18.89</th>
<th>19.34</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r ) (mm)</td>
<td>250</td>
<td>200</td>
<td>150</td>
<td>100</td>
<td>50</td>
<td>0</td>
</tr>
</tbody>
</table>

Distribution of radial and hoop stresses is shown in the figure. Maximum in plane shear stress occurs at the outer radius

\[ \tau_{\text{max}} = \frac{\sigma_c - \sigma_r}{2} = \frac{8.20 - 0}{2} = 4.10 \text{ N/mm}^2 \]
Example 8.3

A thin uniform disc of inner radius 50 mm and outer radius 200 mm is rotating at 6000 rpm about its axis. What are the maximum hoop and radial stresses? Draw the distribution of hoop and radial stresses along the radius of the disc.

Given \( \rho = 7800 \text{kg/m}^3, \ \nu = 0.3, \ N = 6000 \text{rpm} \).

\( R_1, \) inner radius = 50 mm
\( R_2, \) outer radius = 200 mm

Angular speed, \( \omega = \frac{2\omega N}{60} = \frac{2 \times \pi \times 6000}{60} = 628.32 \text{rad/s} \)

Constants, \( k_1 = \frac{3 + \nu}{8} \quad k_2 = \frac{1 + 3\nu}{8} = \frac{1.9}{8} \)

\[ = \frac{3.3}{8} = 0.4125 \quad = 0.2375 \]

Weight density, \( \rho = 7800 \times 9.81 \times 10^{-9} \text{N/mm}^3 = 7.65 \times 10^{-5} \text{N/mm}^3 \)

\[ \frac{\rho k_1 \omega^2}{g} = \frac{7.65 \times 10^{-5} \times 0.4125 \times (628.32)^2}{9810} = 1.27 \times 10^{-3} \text{N/mm}^4 \]

\[ \frac{\rho k_2 \omega^2}{g} = \frac{7.65 \times 10^{-5} \times 0.2375 \times (628.32)^2}{9810} = 0.731 \times 10^{-3} \text{N/mm}^4 \]
Let us calculate also

\[
k_1 \frac{\rho \omega^2}{g} \left( R_1^2 + R_2^2 \right) = 1.27 \times 10^{-3} (50^2 + 200^2) = 53.97 \text{ N/mm}^2
\]

\[
k_1 \frac{\rho \omega^2}{g} \left( R_1^2 R_2^2 \right) = 1.27 \times 10^{-3} (50^2 \times 200^2) = 12.7 \times 10^4 \text{ N}
\]

Radial stress

\[
\sigma_r = \frac{k_1 \rho \omega^2 (R_1^2 + R_2^2)}{g} - k_1 \frac{\rho \omega^2 R_1^2 R_2^2}{r^2} - k_1 \frac{\rho \omega^2 r^2}{g}
\]

\[
= 53.97 - \frac{12.7 \times 10^4}{r^2} - 1.27 \times 10^{-3} r^2
\]

\[
= 53.97 - 50.8 - 3.17 = 0 \quad \text{at} \quad r = 50 \text{ mm}
\]

\[
= 53.97 - 12.7 - 12.7 = 28.57 \text{ N/mm}^2 \quad \text{at} \quad r = 100 \text{ mm}
\]

\[
= 53.97 - 5.64 - 28.57 = 19.76 \text{ N/mm}^2 \quad \text{at} \quad r = 150 \text{ mm}
\]

\[
= 53.97 - 3.17 - 50.80 = 0 \quad \text{at} \quad r = 200 \text{ mm}
\]

Maximum \( \sigma_r \) occurs at

\[
r = \sqrt{R_1 R_2} = \sqrt{50 \times 200} = 100 \text{ mm}
\]

\[
\sigma_r \text{ max} = 28.57 \text{ N/mm}^2
\]
Variation of hoop and radial stresses along the radius of the disc

Circumferential Stress

\[ \sigma_c = k_1 \frac{\rho \omega^2}{g} \left( R_1^2 + R_2^2 \right) + k_1 \frac{\rho \omega^2}{g} \frac{R_1^2 + R_2^2}{r^2} - k_2 \frac{\rho \omega^2 r^2}{g} \]

\[ = 53.97 + \frac{12.7 \times 10^4}{r^2} - 0.731 \times 10^{-3} r^2 \]

\[ = 53.97 + 50.8 - 0.791 \times 10^{-3} \times 2500 \]

\[ = 104.77 - 1.83 = 102.94 \text{ N/mm}^2 \text{ at } r = 50 \text{ mm} \]

\[ = 53.97 + 12.7 - 7.31 = 59.36 \text{ N/mm}^2 \text{ at } r = 100 \text{ mm} \]

\[ = 53.97 + 5.64 - 16.44 = 43.17 \text{ N/mm}^2 \text{ at } r = 150 \text{ mm} \]

\[ = 53.97 + 3.17 - 29.24 = 27.90 \text{ N/mm}^2 \text{ at } r = 200 \text{ mm} \]

The maximum circumferential stress occurs at the inner radius and minimum circumferential stress occurs at the outer radius as is obvious from the figure.
DISC OF UNIFORM STRENGTH

Rotors of steam or gas turbines, on the periphery of which blades are attached, are designed as disc of uniform strength, in which the stress developed due to centrifugal forces are equal and constant independent of the radius. In order to achieve the objective of uniform strength throughout, thickness of the disc is varied along the radius. Consider a disc of radius $R$, rotating at angular speed $\omega$ about its axis, as shown in Figure 3. Take a small element $abcd$ subtending on angle $d\theta$ at the centre of the disc. Disc is of uniform strength. Stress on faces $ab$, $bc$, $cd$, and $da$ of the small element is $\sigma$ as shown. Thickness of the element at radius $r$ is $t$ and say at radius $r + dr$ thickness is $t + dt$.

![Figure 3 Disc of uniform strength](image)
Volume of small element = \( rd\theta tdr \)

Mass of the element = \( \frac{\rho rtd\theta dr}{g} \)

Centrifugal force on small element,

\[
CF = \rho rt \frac{d\theta dr}{g} \times \omega^2 r = \frac{\rho \omega^2 r^2 td\theta dr}{g}
\]

Radial force on face \( ab = rd\theta t\sigma \)

Radial force on face \( cd = (r+dr) d\theta (t+dt)\sigma \)

Forces on faces \( bc \) and \( da = rd\theta t\sigma \)

where \( \sigma \) is the uniform stress.

These forces are inclined at an angle \( \frac{d\theta}{2} \) to horizontal direction as shown.

Resolving all these forces along the radial direction \( OCF \)

\[
\frac{\rho \omega^2 r^2}{g} (tdrd\theta) + \sigma (r + dr)(t + dt)d\theta
\]

\[
= \sigma r d\theta t + 2\sigma t dr \sin \frac{d\theta}{2}
\]
but $d\theta$ is very small, so \[ \sin \frac{d\theta}{2} \approx \frac{d\theta}{2}, \]
in all these terms $d\theta$ is common, simplifying the equation we can write

\[
\frac{\rho \omega^2 r^2 t}{g} dr + \sigma r \, dr + \sigma r \, dt + \sigma dr \, dt = \sigma r \, t + \sigma t \, dr
\]

neglecting $dr \times dt$ as negligible.

or

\[
\frac{\rho \omega^2 r^2 t}{g} dr + \sigma r \, dt = 0
\]

or

\[
\sigma \frac{dt}{t} = -\frac{\rho \omega^2 r}{g} \, dr
\]

or

\[
\frac{dt}{t} = -\frac{\rho \omega^2 r}{g \sigma} \, dr
\]

Integrating this equation, we get

\[
\ln t = -\frac{\rho \omega^2 r^2}{2 \sigma g} + \ln A,
\]
where $\ln A$ is a constant of integration,

\[
\ln \frac{t}{A} = -\frac{\rho \omega^2 r^2}{2\sigma g}
\]

or

\[
t = A e^{-\frac{\rho \omega^2 r^2}{2g\sigma}}
\]

At the center, $r = 0$, thickness $t = t_0$

So, $t_0 = Ae^0 = A$

or constant

$A = t_0$

Equation for thickness becomes

\[
t = t_0 e^{-\frac{\rho \omega^2 r^2}{2g\sigma}}
\]
Example 8.4

A steel disc of a turbine is to be designed so that the radial and circumferential stresses are to be the same throughout the thickness and radius of disc and is equal to 80 MPa, when running at 3500 rpm. If the axial thickness at the centre is 20 mm, what is the thickness at the radius of 500 mm?

$\rho$ for steel = 0.07644N/cm³, $g = 9810$mm/s.

Thickness at centre, $t_0 = 20$ mm
radius $r = 500$ mm

Angular velocity, $\omega = \frac{211 \times 3500}{60} = 366.52$ rad/s
Constant strength, $\sigma = 80$MPa

density = $0.07644 \times 10^{-3}$N/mm³

$$\frac{\rho \omega^2 r^2}{2g\sigma} = \frac{0.07644 \times 10^{-3} \times (366.52)^2 \times 500^2}{2 \times 9810 \times 80}$$

$$= 1.636$$

$$e^{-1.636} = 0.1947$$

Thickness, $t = t_0 \ e^{-1.636} = 20 \times 0.1947 = 3.895$ mm
STRESSES IN ROTATING LONG CYLINDERS

The analysis of stresses in long cylinders is similar to that of a thin disc. The only difference is that length of the cylinder along the axis of rotation is large as compared to its radius and axial stress is also taken into account, i.e. at any radius \( r \) of the cylinder there are three stresses, i.e. \( \sigma_a \), axial stress; \( \sigma_r \) radial stress, and \( \sigma_c \) the circumferential stress. While developing the theory for long rotating cylinders, the following assumptions are made:

1. Transverse sections of the cylinder remain plane at high speeds of rotation. This is true only for sections away from the ends.

2. At the central cross section of the cylinder, shear stress is zero due to symmetry and there are three principal stresses, i.e. \( \sigma_r \), \( \sigma_a \), and \( \sigma_c \) as stated above.
If $E$ is the Young’s modulus and $\nu$ is the Poisson’s ratio, then,

Hoop strain, 
\[ \varepsilon_c = \frac{\sigma_c}{E} - \frac{\nu \sigma_r}{E} - \frac{\nu \sigma_a}{E} = \frac{1}{E} \left[ \sigma_c - \nu(\sigma_r + \sigma_a) \right] \]

Radial strain, 
\[ \varepsilon_r = \frac{\sigma_r}{E} - \frac{\nu \sigma_c}{E} - \frac{\nu \sigma_a}{E} = \frac{1}{E} \left[ \sigma_r - \nu(\sigma_c + \sigma_a) \right] \]

Axial strain, 
\[ \varepsilon_a = \frac{\sigma_a}{E} - \frac{\nu \sigma_r}{E} - \frac{\nu \sigma_c}{E} = \frac{1}{E} \left[ \sigma_a - \nu(\sigma_c + \sigma_r) \right] \]

We have derived expression for $(\sigma_c - \sigma_r)$ for a small element subtending an angle $d\theta$ at the centre of disc, the equations of equilibrium give

\[ \sigma_c - \sigma_r = r \frac{d\sigma_r}{dr} + \frac{\rho \omega^2 r^2}{\sigma} \]

But circumferential strain,
\[ \varepsilon_c = \frac{u}{r} = \frac{1}{E} \left[ \sigma_c - \nu(\sigma_r + \sigma_a) \right] \]

Radial strain,
\[ \varepsilon_r = \frac{du}{dr} = \frac{1}{E} \left[ \sigma_r - \nu(\sigma_c + \sigma_a) \right] \]
Differentiating \( \varepsilon_c = \frac{u}{r} = \frac{1}{E} [\sigma_c - \nu(\sigma_r + \sigma_a)] \) with respect to \( r \), we get

\[
\frac{du}{dr} = \frac{1}{E} \left[ \sigma_c - \nu(\sigma_r + \sigma_a) \right] + \frac{1}{E} \left[ r \frac{d\sigma_c}{dr} - \nu r \left( \frac{d\sigma_r}{dr} + \frac{d\sigma_a}{dr} \right) \right]
\]

Equating \( \varepsilon_r = \frac{du}{dr} = \frac{1}{E} [\sigma_r - \nu(\sigma_c + \sigma_a)] \) and the above equation and after simplification, we get

\[
\sigma_r (1 + \nu) - \sigma_c (1 + \nu) - r \frac{d\sigma_c}{dr} + r\nu \frac{d\sigma_r}{dr} + r\nu \frac{d\sigma_a}{dr} = 0
\]

As per the first assumption, the transverse sections remain plane after the long cylinder starts rotating at high angular speeds; it is implied that the axial strain is constant, therefore,

\[
\varepsilon_a = \frac{1}{E} [\sigma_a - \nu(\sigma_c + \sigma_2)] = \text{a constant}
\]

In this \( E \) and \( \nu \) are elastic constants, therefore

\[
\sigma_a - \nu(\sigma_c + \sigma_r) = \text{a constant}
\]

Differentiating the above equation with respect to \( r \), we get

\[
\frac{d\sigma_a}{dr} - \nu \frac{d\sigma_c}{dr} - \nu \frac{d\sigma_r}{dr} = 0
\]
Substituting the value of \( r \frac{d\sigma_a}{dr} \) in \( \sigma_r(1+v) - \sigma_c(1+v) - r \frac{d\sigma_c}{dr} + rv \frac{d\sigma_r}{dr} + rv^2 \frac{d\sigma_c}{dr} + rv^2 \frac{d\sigma_r}{dr} = 0 \)

or

\[ -(\sigma_c - \sigma_r)(1+v) - r(1-v^2) \frac{d\sigma_c}{dr} + rv(1+v) \frac{d\sigma_r}{dr} = 0 \]

After further simplification

\[ (\sigma_c - \sigma_r) + r(1-v) \frac{d\sigma_c}{dr} - rv \frac{d\sigma_r}{dr} = 0 \]

But from \( \sigma_c - \sigma_r = r \frac{d\sigma_r}{dr} + \frac{\rho \omega^2 r^2}{g} \), we get

\[ \sigma_c - \sigma_r = r \frac{d\sigma_r}{dr} + \frac{\rho \omega^2 r^2}{g} \]

Substituting this value of \( (\sigma_c - \sigma_r) \) in \( (\sigma_c - \sigma_r) + r(1-v) \frac{d\sigma_c}{dr} - rv \frac{d\sigma_r}{dr} = 0 \)

\[ r \frac{d\sigma_r}{dr} + \frac{\rho \omega^2 r^2}{g} + r(1-v) \frac{d\sigma_c}{dr} - rv \frac{d\sigma_r}{dr} = 0 \]

\[ r(1-v) \frac{d\sigma_r}{dr} + r(1-v) \frac{d\sigma_c}{dr} + \frac{\rho \omega^2 r^2}{g} = 0 \]
or
\[
\frac{d\sigma_r}{dr} + \frac{d\sigma_c}{dr} = -\frac{\rho \omega^2 r}{(1-v)g}
\]

Integrating this, we get
\[
\sigma_c + \sigma_r = -\frac{\rho \omega^2 r^2}{2(1-v)g} + A,
\]
where \(A\) is constant of integration.

But,
\[
\sigma_c - \sigma_r = \frac{r d\sigma_r}{dr} + \frac{\rho \omega^2 r^2}{g}
\]

From \(\sigma_c - \sigma_r = \frac{r d\sigma_r}{dr} + \frac{\rho \omega^2 r^2}{g}\) and \(\sigma_c - \sigma_r = \frac{r d\sigma_r}{dr} + \frac{\rho \omega^2 r^2}{g}\), we get
\[
2\sigma_r = -\frac{\rho \omega^2 r^2}{2(1-v)g} + A - r \frac{d\sigma_r}{dr} - \frac{\rho \omega^2 r^2}{g}
\]
\[
2\sigma_r + r \frac{d\sigma_r}{dr} = A - \frac{\rho \omega^2 r^2}{2g} \left(\frac{3-2v}{1-v}\right)
\]

Multiplying throughout by \(r\), we get
\[
2r\sigma_r + r^2 \frac{d\sigma_r}{dr} = Ar - \frac{\rho \omega^2 r^3}{2g} \left(\frac{3-2v}{1-v}\right)
\]
Integrating this equation, we get

\[ \sigma_r r^2 = \frac{A r^2}{2} - \frac{\rho \omega^2 r^4}{8g} \left( \frac{3-2v}{1-v} \right) + B \]

where \( B \) is another constant of integration.

Or radial stress,

\[ \sigma_r = -\frac{A}{2} + \frac{B}{r^2} - \frac{\rho \omega^2 r^2}{8g} \left( \frac{3-2v}{1-v} \right) \]

But from \( \sigma_c - \sigma_r = r \frac{d\sigma_r}{dr} + \frac{\rho \omega^2 r^2}{g} \),

\[ \sigma_c + \sigma_r = \frac{-\rho \omega^2 r^2}{2(1-v)g} + A \]

From Eqs \( \sigma_c - \sigma_r = r \frac{d\sigma_r}{dr} + \frac{\rho \omega^2 r^2}{g} \) and \( \sigma_r = -\frac{A}{2} + \frac{B}{r^2} - \frac{\rho \omega^2 r^2}{8g} \left( \frac{3-2v}{1-v} \right) \), we get the value of \( \sigma_c \)

\[ \sigma_c = -\frac{\rho \omega^2 r^2}{2(1-v)g} + A - \frac{A}{2} - \frac{B}{r^2} + \frac{\rho \omega^2 r^2}{8g} \left( \frac{3-2v}{1-v} \right) \]

or hoop stress,

\[ \sigma_c = \frac{A}{2} - \frac{B}{r^2} - \frac{\rho \omega^2 r^2}{8g} \left( \frac{1+2v}{1-v} \right) \]
Solid cylinder

In the case of solid long cylinder, stress at the centre cannot be infinite; therefore constant \( B = 0 \), expression for stresses will be

\[
\sigma_r = \frac{A}{2} - \frac{\rho \omega^2 r^2}{8g} \left( \frac{3 - 2v}{1 - v} \right)
\]

\[
\sigma_c = \frac{A}{2} - \frac{\rho \omega^2 r^2}{8g} \left( \frac{1 + 2v}{1 - v} \right)
\]

At the outer free surface, radial stress \( \sigma_r \) has to be zero, so

at \( r = R \), outerradius \( \sigma_r = 0 \),

\[
0 = \frac{A}{2} - \frac{\rho \omega^2 R^2}{8g} \left( \frac{3 - 2v}{1 - v} \right)
\]

\[
\therefore \quad \frac{A}{2} = \frac{\rho \omega^2 R^2}{8g} \left( \frac{3 - 2v}{1 - v} \right)
\]

or constant
Expression for stresses will now be

\[
\sigma_r = \frac{\rho \omega^2}{8g} \times \left(3\frac{2\nu}{1-\nu}\right)\left[R^2 - r^2\right]
\]

\[
\sigma_c = \frac{\rho \omega^2 R^2}{8g} \left(3\frac{2\nu}{1-\nu}\right) - \frac{\rho \omega^2 r^2}{8g} \left(\frac{1+2\nu}{1-\nu}\right)
\]

If we put constant

\[
\frac{3-2\nu}{8(1-\nu)} = k_3 \quad \text{and} \quad \frac{1+2\nu}{8(1-\nu)} = k_4
\]

Stresses will be

\[
\sigma_r = \frac{\rho \omega^2}{g} k_3 \left(R^2 - r^2\right)
\]

\[
\sigma_c = \frac{\rho \omega^2}{g} \left[k_3 R^2 - k_4 r^2\right]
\]

\(\sigma_r\) and \(\sigma_c\) are maximum at the centre of the cylinder

\[
\sigma_{r,\max} = \sigma_{c,\max} = \frac{\rho \omega^2 k_3 R^2}{g}
\]
**Hollow cylinder**

Stresses in a hollow cylinder are

radial stress,
\[ \sigma_r = \frac{A}{2} + \frac{B}{r^2} - k_3 \frac{\rho \omega^2 r^2}{g} \]

hoop stress,
\[ \sigma_c = \frac{A}{2} - \frac{B}{r^2} - k_4 \frac{\rho \omega^2 r^2}{g} \]

But radial stress is zero at inner and outer free surfaces of cylinder, i.e.

\[ \sigma_r^\pm = 0 \text{ at } r = R_1, \text{ inner radius} \]
\[ r = R_2, \text{ outer radius} \]

So,
\[ 0 = \frac{A}{2} + \frac{B}{R_1^2} - k_3 \frac{\rho \omega^2 R_1^2}{g} \]
\[ 0 = \frac{A}{2} + \frac{B}{R_2^2} - k_3 \frac{\rho \omega^2 R_2^2}{g} \]

From these equations
\[ \frac{A}{2} = +k_3 \frac{\rho \omega^2}{g} \left( R_1^2 + R_2^2 \right) \]
\[ B = -k_3 \frac{\rho \omega^2}{g} \left( R_1 R_2^2 \right) \]
Finally the expressions for stresses are

\[
\sigma_r = k_3 \frac{\rho \omega^2}{g} \left[ \left( R_1^2 + R_2^2 \right) - \frac{R_1^2 R_2^2}{r^2} - r^2 \right]
\]

\[
\sigma_c = k_3 \frac{\rho \omega^2}{g} \left( R_1^2 + R_2^2 \right) + k_3 \frac{\rho \omega^2}{g} \frac{R_1^2 R_2^2}{r^2} - k_4 \frac{\rho \omega^2}{g} r^2
\]

Obviously \( \sigma_c \) will be maximum when \( r \) is minimum, i.e. at inner radius, \( R_1 \)

\[
\sigma_{c_{\text{max}}} = k_3 \frac{\rho \omega^2}{g} \left( R_1^2 + R_2^2 \right) + k_3 \frac{\rho \omega^2}{g} \frac{R_1^2 R_2^2}{R_1^2} - k_4 \frac{\rho \omega^2}{g} \times R_1^2
\]

\[
= \frac{\rho \omega^2}{g} \left[ k_3 \left( 2R_2^2 + R_1^2 \right) - k_4 R_1^2 \right]
\]

To obtain the value of \( \sigma_{r_{\text{max}}} \), let us put

\[
\frac{d \sigma_r}{dr} = 0 = k_3 \frac{\rho \omega^2}{g} \left[ \frac{2R_1^2 R_2^2}{r^3} - 2r \right]
\]

or

\[
r^4 = R_1^2 R_2^2
\]

\[
r = \sqrt{R_1 R_2}
\]
Putting this value of $r$ in expression for $\sigma_r$

\[
\sigma_{r_{\text{max}}} = \frac{k_3 \rho \omega^2}{g} \left[ R_1^2 + R_2^2 - \frac{R_1^2 R_2^2}{R_1 R_2} - R_1 R_2 \right]
\]

\[
= \frac{k_3 \rho \omega^2}{g} \left[ (R_2 - R_1)^2 \right]
\]
Example 8.5

A solid long cylinder of diameter 600 mm is rotating at 3000 rpm. Calculate (i) maximum and minimum hoop stresses and (ii) maximum radial stress.

Given \( \rho = 0.07644 \text{ N/cm}^3, \ g = 9.8 \text{ m/s}^2, \ \nu = 0.3 \)

\[ R = 300 \text{ mm} \]
\[ \rho = 0.07644 \times 10^{-3} \text{ N/mm}^3 \]
\[ g = 9800 \text{ mm/s}^2, \ \nu = 0.3 \]

\[ \omega = \frac{2\pi N}{60} = \frac{2\pi \times 3000}{60} = 314.16 \text{ rad/s} \]
\[ k_3 = \frac{(3 - 2\nu)}{8(1 - \nu)} = \frac{3 - 2 \times 0.3}{8(1 - 0.3)} = 0.4286 \]
\[ k_4 = \frac{(1 + 2\nu)}{8(1 - \nu)} = \frac{1 + 2 \times 0.3}{8(1 - 0.3)} = 0.2857 \]
\[ \frac{\rho \omega^2}{g} = \frac{0.07644 \times 10^{-3} \times (314.16)^2}{9800} = 7.698 \times 10^{-4} \]

\[ \sigma_{r_{\text{max}}} = \sigma_{c_{\text{max}}} = k_3 \frac{\rho \omega^2}{g} \times R^2 = 0.4286 \times 7.698 \times 10^{-4} \times 300^2 \]

\[ = 29.694 \text{ N/mm}^2 \]
Minimum hoop stress occurs at outer radius

\[ \sigma_c = \frac{\rho \omega^2}{g} \left[ k_3 R^2 - k_4 R^2 \right] = \frac{\rho \omega^2}{g} (k_3 - k_4) R^2 \]

\[ = 7.698 \times 10^{-4} \left[ 0.4286 - 0.2837 \right] \times 300^2 \]

\[ = 9.9 \text{ N/mm}^2 \]
Example 8.6

A long cylinder of inside diameter 50 mm and outside diameter 250 mm is rotating at 5000 rpm, calculate (i) minimum and maximum hoop stresses and (ii) maximum radial stress and at what radius it occurs?

Given \( \rho = 0.07644 \times 10^{-3}/\text{mm}^3, \quad \nu = 0.3, \quad g = 9.8 \text{ m/s}^2 \)

Inner radius, \( R_1 = 25 \text{ mm} \)

Outer radius, \( R_2 = 125 \text{ mm} \).

Constants

\[
k_3 = \frac{3 - 2\nu}{8(1 - \nu)} = \frac{3 - 0.6}{8(1 - 0.7)} = 0.4286
\]

\[
k_4 = \frac{1 + 2\nu}{8(1 - \nu)} = \frac{1.6}{5.6} = 0.2857
\]

Angular speed,

\[
\omega = \frac{2\pi \times 5000}{60} = 523.6 \text{ rad/sec.}
\]

\[
\frac{\rho \omega^2}{g} = \frac{0.07644 \times 10^{-3}(523.6)^2}{9800} = 2.1384 \times 10^{-3} \text{ N/mm}^4
\]
\( \sigma_{r_{\text{max}}} \)

\[
r = \sqrt{R_1 R_2} = \sqrt{25 \times 125} = 55.90 \text{ mm}
\]

\[
\sigma_{r_{\text{max}}} = k_3 \frac{\rho \omega^2}{g} (R_2 - R_1)^2 = 0.4286 \times 2.1384 \times 10^{-3} (125 - 25)^2
\]

\[
= 9.165 \text{ N/mm}^2
\]

Circumferential stress,

\[
\sigma_c = k_3 \frac{\rho \omega^2}{g} \left( R_1^2 + R_2^2 \right) + k_3 \frac{\rho \omega^2}{g} \frac{R_1^2 R_2^2}{r^2} - k_4 \frac{\rho \omega^2 r^2}{g}
\]

\[
= \frac{\rho \omega^2}{g} \left[ k_3 \left( R_1^2 + R_2^2 \right) + k_3 \frac{R_1^2 R_2^2}{r^2} - k_4 \cdot r^2 \right]
\]

At the inner radius,

\[
r = R_1,
\]

\[
\sigma_{c_{\text{max}}} = \frac{\rho \omega^2}{g} \left[ k_3 \left( 2 R_2^2 + R_1^2 \right) - k_4 \cdot R_1^2 \right]
\]

\[
= 2.1384 \times 10^{-3} \left[ 0.4286 \left( 2 \times 125^2 + 25^2 \right) - 0.2857 \times 25^2 \right]
\]

\[
= 2.1384 \times 10^{-3} \left[ 13661.625 - 178.5625 \right]
\]

\[
= 28.83 \text{ N/mm}^2
\]
At the outer radius, $\sigma_{c_{\text{min}}}$ at $r = R_2$

$$\sigma_{c_{\text{min}}} = \frac{\rho \omega^2}{g}\left[k_3 \left(2R_1^2 + R_2^2\right) - k_4 \cdot R_2^2\right]$$

$$= 2.1384 \times 10^{-3} \left[0.4286 \left(2 \times 25^2 + 125^2\right) - 0.2857 \times 125^2\right]$$

$$= 2.1384 \times 10^{-3} \left[7232.625 - 4464.0625\right]$$

$$= 5.92 \text{ N/mm}^2$$
Exercise 8.1

A solid long cylinder of diameter 400 mm is rotating about its axis at an angular speed of 300 rad/s. Determine the maximum and minimum values of radial and hoop stresses.

Given $\rho = 0.07644 \times 10^3$ N/mm$^3$, $\nu =0.3$, $g = 9.81$m/s$^2$
Exercise 8.2

A long cylinder of steel of outer diameter 750 mm and inner diameter 250 mm is rotating about its axis at 4000 rpm. Determine the radial stress and circumferential stress along the radius of cylinder.

Given $\rho = 0.078 \text{ N/cm}^3$, $\nu = 0.3$, $g = 980 \text{ m/s}^2$
THERMAL STRESSES IN A THIN DISC

Consider a thin disc rotating at a high angular speed $\omega$ and subjected to temperature variation at the same time as shown in Figure 4. Say the stresses are $\sigma_c$, circumferential, and $\sigma_r$, radial, and $\alpha$ is the coefficient of thermal expansion of the material, $T$ is the change in temperature. Consider a small element $abcd$ subtending an angle $d\theta$ at the centre, radial thickness $dr$. At high speed

$r \rightarrow$ changes to $r + u$
\[dr \rightarrow$ changes to $dr + du$\[Circumferential strain,\]

$$\varepsilon_c = \frac{r}{u} = \frac{\sigma_c}{F} - v \frac{\sigma_r}{F} + \alpha T$$

Figure 4 Rotating thin disc
Radial strain,

\[ \varepsilon_r = \frac{du}{dr} = \frac{\sigma_r}{E} - v \frac{\sigma_c}{E} + \alpha T = \frac{du}{dr} \]

The relationship between the circumferential and radial stresses (page 9):

\[ \sigma_c - \sigma_r = r \frac{d\sigma_r}{dr} + \frac{\rho \omega^2 r^2}{g} \]

From

\[ \varepsilon_c = \frac{u}{r} = \frac{\sigma_c}{E} - v \frac{\sigma_r}{E} + \alpha T \]

\[ \frac{du}{dr} = \left( \frac{\sigma_c}{E} - v \frac{\sigma_r}{E} + \alpha T \right) + r \left[ \frac{d\sigma_c}{dr} - v \frac{d\sigma_r}{dr} + E \alpha \frac{dT}{dr} \right] \]

Equating \( \varepsilon_r = \frac{du}{dr} = \frac{\sigma_r}{E} - v \frac{\sigma_c}{E} + \alpha T = \frac{du}{dr} \) and the above equation,

\[ \frac{\sigma_r}{E} - v \frac{\sigma_c}{E} + \alpha T = \frac{\sigma_c}{E} - v \frac{\sigma_r}{E} + \alpha T + r \left[ \frac{d\sigma_c}{dr} - v \frac{d\sigma_r}{dr} + E \alpha \frac{dT}{dr} \right] \]

or

\[ (\sigma_c - \sigma_r)(1+v) = -r \frac{d\sigma_c}{dr} + rv \frac{d\sigma_r}{dr} - Er \alpha \frac{dT}{dr} \]
Putting the value of \((\sigma_c - \sigma_r)\) from

\[
\sigma_c - \sigma_r = r \frac{d\sigma_r}{dr} + \frac{\rho \omega^2 r^2}{g} + \frac{\rho \omega^2 r^2}{g}
\]

\[
(1 + v) \frac{d\sigma_r}{dr} + (1 + v) \frac{\rho \omega^2 r}{g} = -r \frac{d\sigma_c}{dr} + rv \frac{d\sigma_r}{dr} - Er\alpha \frac{dT}{dr}
\]

\[
\frac{d\sigma_r}{dr} + \frac{d\sigma_c}{dr} = -(1 + v) \frac{\rho \omega^2 r}{g} - E\alpha \frac{dT}{dr}
\]

Integrating this equation

\[
\sigma_r + \sigma_c = -(1 + v) \frac{\rho \omega^2 r^2}{2g} - E\alpha T + A
\]

where \(A\) is a constant of integration.

From \(\sigma_c - \sigma_r = r \frac{d\sigma_r}{dr} + \frac{\rho \omega^2 r^2}{g}\) and the above eq., eliminating \(\sigma_c\)

\[
2\sigma_r = -(1 + v) \frac{\rho \omega^2 r^2}{2g} - E\alpha T + A - r \frac{d\sigma_r}{dr} - \frac{\rho \omega^2 r^2}{g}
\]

\[
2\sigma_r + r \frac{d\sigma_r}{dr} = -(1 + v) \frac{\rho \omega^2 r^2}{2g} - E\alpha T + A
\]

Multiplying throughout by \(r\)

\[
2r\sigma_r + r^2 \frac{d\sigma_r}{dr} = -(1 + v) \frac{\rho \omega^2 r^3}{2g} - E\alpha Tr + Ar
\]
Integrating this, we get

\[ r^2 \sigma_r = -(3+\nu) \frac{\rho \omega^2 r^4}{8g} - E\alpha \int Tr \, dr + \frac{Ar^2}{2} + B \]

where B is another constant of integration.

Radial stress,

\[ \sigma_r = \frac{A}{2} + \frac{B}{r^2} - \frac{3+\nu}{8} \frac{\rho \omega^2 r^2}{g} - \frac{E\alpha}{r^2} \int Tr \, dr \]

putting the value of \( \sigma_r \) in \( \sigma_r + \sigma_c = -(1+\nu) \frac{\rho \omega^2 r^2}{2g} - E\alpha T + A \), we get

\[ \sigma_c = \frac{A}{2} - \frac{B}{r^2} - \frac{1+3\nu}{8} \times \frac{\rho \omega^2 r^2}{g} - E\alpha T + \frac{E\alpha}{r^2} \int Tr \, dr \]

Constants A and B can be determined by using the boundary conditions.

For a solid disc, constant B = 0, because stress cannot be infinite at the centre of the disc.
Example 8.7

A thin disc of outer radius 300 mm and inner radius 100 mm is rotating about its axis at 3500 rpm. Temperature in the disc has a linear variation of temperature with temperature 80°C at outer radius and 20°C at inner radius, calculate the maximum stress.

\[ E = 208 \text{ kN/mm}^2, \quad \rho = 0.07644 \text{ N/cm}^3; \quad \nu = 0.3, \]

\[ g = 9.80 \text{ m/s}^2 \]

Given \[ \alpha = 11 \times 10^{-6}/\text{°C} \]

Angular speed,

\[ \omega = \frac{2\pi \times 3500}{60} = 366.52 \text{ rad/s} \]

\[ \sigma_r = \frac{A}{2} + \frac{B}{r^2} - \frac{3 + \nu}{8} \times \frac{\rho \omega^2 \nu^2}{g} - \frac{E \alpha}{r^2} \int T_r \, dr \]

Temperature variation

\[ T = 20 + 0.3(r - 100) \]

at \[ r = 100 \text{ mm}, \quad T = 20^\circ \text{C} \]

\[ r = 300 \text{ mm}, \quad T = 80^\circ \text{C} \]

Radial stress, \( \sigma_r = 0 \) at \( r = 100 \text{ mm}, \ r = 300 \text{ mm} \)
Taking temperature $20^\circ$C, constant from center to inner radius of disc
Putting second boundary condition,

\[
0 = \frac{A}{2} + \frac{B}{300^2} - \frac{3 + 0.3 \times \rho \omega^2 \times 300^2}{8} - \frac{E \alpha}{300^2} \int_{100}^{300} [20r + 0.3r(r - 100)] \, dr
\]

\[
0 = \frac{A}{2} + \frac{B}{90000} - \frac{3.3}{8} \times \frac{0.07644 \times 10^{-3} \times (366.52)^2 \times 300^2}{9800} - \frac{E \alpha}{90000} \int_{100}^{300} (0.3r^2 - 10r) \, dr
\]

\[
0 = \frac{A}{2} + \frac{B}{90000} - 38.88 - \frac{208000 \times 11 \times 10^{-6}}{90000} \int_{100}^{300} (0.3r^2 - 10r) \, dr
\]

\[
0 = \frac{A}{2} + \frac{B}{90000} - 38.88 - 0.2542 \times 10^{-4} \left| 0.1r^3 - 5r^2 \right|_{100}^{300}
\]

\[
= \frac{A}{2} + \frac{B}{90000} - 38.88 - 0.2542 \times 10^{-4} [(2700000 - 450000) - (100000 - 50000)]
\]

\[
= \frac{A}{2} + \frac{B}{90000} - 38.88 - 0.2542 \times 10^{-4} [2250000 - 50000]
\]

\[
= \frac{A}{2} + \frac{B}{90000} - 38.88 - 0.2542 \times 10^{-4} \times 2200000
\]

\[
= \frac{A}{2} + \frac{B}{90000} - 38.88 - 55.924
\]

\[
\frac{A}{2} + \frac{B}{90000} = 94.804
\]
From Eqs (i) and (ii)

\[
\frac{B}{10,000} - \frac{B}{90000} = -67.604
\]

\[
B - \frac{B}{9} = -67.604 \times 10^4
\]

\[
B = -76.0545 \times 10^4
\] (iii)

\[
\frac{A}{2} = 94.804 - \frac{B}{90000} = 94.804 + 8.4505
\]

\[
\frac{A}{2} = 103.2545
\]

Maximum stress occurs at the inner radius at \( r = 100 \text{ mm} \).
maximum stress i.e. circumferential stress occurs at the inner radius.