A sequence of real numbers is a real-valued function whose domain is the set of positive integers, that is, it is a function \( f : \mathbb{Z}^+ \to \mathbb{R} \). We denote it by \( (a_n)_{n=1}^\infty \) by taking \( a_n = f(n) \) for every \( n \in \mathbb{Z}^+ \).

### 2. Biographies of the following mathematicians (and scientists):

- (a) Nicole Oresme (1320-1382)
- (b) Richard Dedekind (1831-1916)
- (c) Albert of Saxony (1316-1390)
- (d) Brook Taylor (1685-1731)
- (e) Colin Maclaurin (1698-1746)

### Write ‘TRUE’ or ‘FALSE’ for each of the following statements. If you claim ‘TRUE’, prove that. If you claim ‘FALSE’, give a counter example and give a corresponding statement that becomes true if you correct some hypothesis.

1. FALSE A sequence of real numbers is a real-valued function whose domain is \( \mathbb{R} \).
   **Answer:** A sequence of real numbers is a real-valued function whose domain is the set of positive integers, that is, it is a function \( f : \mathbb{Z}^+ \to \mathbb{R} \). We denote it by \( (a_n)_{n=1}^\infty \) by taking \( a_n = f(n) \) for every \( n \in \mathbb{Z}^+ \).

2. FALSE The set \( \mathbb{R}^+ \) of all positive real numbers has a smallest element.
   **Answer:** Suppose for the contrary that \( \mathbb{R}^+ \) has a smallest element \( a \). That is, \( a \) is a positive real number and for every positive real number \( c, \ a < c \). Since \( a \) is positive, \( a/2 \) would be positive and strictly smaller than \( a \) which contradicts with \( a \) being the smallest positive real number. This contradiction shows that \( \mathbb{R}^+ \) has no smallest element.

3. FALSE A nonempty subset \( A \) of \( \mathbb{R} \) is said to be bounded from above if there exists \( M \in \mathbb{R} \) such that for all \( a \in A \).
   **Answer:** A nonempty subset \( A \) of \( \mathbb{R} \) is said to be bounded from above if there exists \( M \in \mathbb{R} \) such that for all \( a \in A \).

4. FALSE A nonempty subset \( A \) of \( \mathbb{R} \) is said to be bounded if there exists \( M \in \mathbb{R} \) such that for all \( a \in A \).
   **Answer:** This is the definition of being bounded from above: A nonempty subset \( A \) of \( \mathbb{R} \) is said to be bounded from above if there exists \( M \in \mathbb{R} \) such that for all \( a \in A \). Similarly, a nonempty subset \( A \) of \( \mathbb{R} \) is said to be bounded from below if there exists \( m \in \mathbb{R} \) such that for all \( a \in A \). We then say that a nonempty subset \( A \) of \( \mathbb{R} \) is said to be bounded if it is bounded from above and from below, that is, if there exist real numbers \( m \) and \( M \) such that \( m \leq a \leq M \) for all \( a \in A \).

5. FALSE A real number \( L \) is not a lower bound of a nonempty subset \( A \) of \( \mathbb{R} \) if there exists \( a \in A \) such that \( a > L \).
   **Answer:** A real number \( L \) is a lower bound of a nonempty subset \( A \) of \( \mathbb{R} \) if there exists \( a \in A \) such that \( a < L \). A real number \( L \) is a lower bound of a nonempty subset \( A \) of \( \mathbb{R} \) if there exists \( a \in A \) such that \( a < L \).

6. TRUE A nonempty subset \( A \) of \( \mathbb{R} \) is bounded if and only if there exists a real number \( K > 0 \) such that \( |a| \leq K \) for all \( a \in A \).
   **Answer:** If a nonempty subset \( A \) of \( \mathbb{R} \) is bounded, then it is bounded from above and from below, that is, there exist real numbers \( m \) and \( M \) such that \( m \leq a \leq M \) for all \( a \in A \). Let \( K = \max(||m| + 1, |M| + 1|) \). Then \( K > 0 \) and for all \( a \in A \),

\[
-K \leq -(|m| + 1) \leq -|m| \leq m \leq a \leq M \leq |M| \leq |M| + 1 \leq K,
\]

which means \( |a| \leq K \) for all \( a \in A \). Conversely, if there exists a real number \( K > 0 \) such that \( |a| \leq K \) for all \( a \in A \), then \( -K \leq a \leq K \) for all \( a \in A \) which means that \( A \) is bounded from below (by \(-K\)) and bounded from above (by \(K\)), that is, \( A \) is a bounded set of real numbers.

7. TRUE \( \inf(\mathbb{R}^+) = 0 \), that is, 0 is the greatest lower bound of the set \( \mathbb{R}^+ \) of all positive real numbers.
   **Answer:** Clearly 0 is a lower bound for \( \mathbb{R}^+ \). It is the greatest lower bound of \( \mathbb{R}^+ \) because if \( a > 0 \) is a real number, then \( a \) is not a lower bound for \( \mathbb{R}^+ \) since \( a > a/2 \) and \( a/2 \in \mathbb{R}^+ \). So 0 is the greatest lower bound of \( \mathbb{R}^+ \).
8. FALSE The set $\mathbb{Z}^+$ of positive integers is bounded from above.  
Answer: By the Archimedean Property of $\mathbb{R}$, the set $\mathbb{Z}^+$ of all positive integers is not bounded from above.

9. FALSE There exists a real number $M$ such that $n \leq M$ for every positive integer $n$.  
Answer: By the Archimedean Property of $\mathbb{R}$, the set $\mathbb{Z}^+$ of all positive integers is not bounded from above, that is, for every $M \in \mathbb{R}$, $M$ is not an upper bound of $\mathbb{Z}^+$ which means that there exists $n \in \mathbb{Z}^+$ such that $n > M$.

10. TRUE For each real number $a > 0$ and for each real number $b$, there exists $n \in \mathbb{Z}^+$ such that $na > b$.  
Answer: This is the Archimedean Principle. See the lecture notes for its proof.

11. FALSE There exists a real number $\epsilon > 0$ such that $\frac{1}{n} \geq \epsilon$ for every $n \in \mathbb{Z}^+$.  
Answer: By the Archimedean Principle, for each real number $\epsilon > 0$, since $\frac{1}{\epsilon}$ is a real number, there exists $n \in \mathbb{Z}^+$ greater than that real number $\frac{1}{\epsilon}$, that is, such that $n > \frac{1}{\epsilon}$ which gives $\frac{1}{n} < \epsilon$.

12. FALSE A sequence $(a_n)_{n=1}^\infty$ of real numbers is said to be bounded if there exists a real number $M$ such that $a_n \leq M$ for all $n \in \mathbb{Z}^+$.  
Answer: This is the definition of being bounded from above: A sequence $(a_n)_{n=1}^\infty$ of real numbers is said to be bounded from above if there exists a real number $M$ such that $a_n \leq M$ for all $n \in \mathbb{Z}^+$. Similarly, a sequence $(a_n)_{n=1}^\infty$ of real numbers is said to be bounded from below if there exists a real number $m$ such that $a_n \geq m$ for all $n \in \mathbb{Z}^+$. We then say that a sequence $(a_n)_{n=1}^\infty$ is bounded if it is bounded from above and below, that is, if there exists real numbers $m$ and $M$ such that $m \leq a_n \leq M$ for all $n \in \mathbb{Z}^+$.

13. TRUE A sequence $(a_n)_{n=1}^\infty$ of real numbers is bounded if and only if there exists a real number $K > 0$ such that $|a_n| \leq K$ for all $n \in \mathbb{Z}^+$.  
Answer: If a sequence $(a_n)_{n=1}^\infty$ is bounded, then it is bounded from above and below, that is, there exists real numbers $m$ and $M$ such that $m \leq a_n \leq M$ for all $n \in \mathbb{Z}^+$. Let $K = \max(|m| + 1, |M| + 1)$. Then for all $n \in \mathbb{Z}^+$, 
\[-K \leq -(|m| + 1) = -|m| \leq m \leq a_n \leq M \leq |M| \leq |M| + 1 \leq K,\]
so that 
\[-K \leq -a_n \leq K\]
which means $|a_n| \leq K$ for all $n \in \mathbb{Z}^+$. Conversely, if there exists a real number $K > 0$ such that $|a_n| \leq K$ for all $n \in \mathbb{Z}^+$, then $-K \leq a_n \leq K$ for all $n \in \mathbb{Z}^+$ which means that $(a_n)_{n=1}^\infty$ is bounded from below (by $-K$) and bounded from above (by $K$), that is, $(a_n)_{n=1}^\infty$ is a a bounded sequence.

14. TRUE $\inf \left\{ \frac{1}{n} \mid n \in \mathbb{Z}^+ \right\} = 0$, that is, 0 is the greatest lower bound of the sequence $\left( \frac{1}{n} \right)_{n=1}^\infty$.  
Answer: Clearly 0 is a lower bound for the sequence $\left( \frac{1}{n} \right)_{n=1}^\infty$. It is the greatest lower bound of the sequence $\left( \frac{1}{n} \right)_{n=1}^\infty$ because if $\epsilon > 0$ is a real number, then by the Archimedean Principle, there exists a positive integer $N$ such that $N > \frac{1}{\epsilon}$ and so $\frac{1}{N} < \epsilon$ which implies that $\epsilon > 0$ is not a lower bound of the sequence $\left( \frac{1}{n} \right)_{n=1}^\infty$.

15. FALSE The set $\mathbb{R}^-$ of all negative real numbers has a greatest element.  
Answer: Suppose for the contrary that $\mathbb{R}^-$ has a greatest element $a$. That is, $a$ is a negative real number and for every negative real number $b$, $a \geq b$. Since $a$ is negative, $a/2$ would be negative and strictly greater than $a$ which contradicts with $a$ being the greatest negative real number. This contradiction shows that $\mathbb{R}^-$ has no greatest element.

16. TRUE $\sup \left\{ \frac{-1}{n} \mid n \in \mathbb{Z}^+ \right\} = 0$, that is, 0 is the least upper bound of the sequence $\left( \frac{-1}{n} \right)_{n=1}^\infty$.  
Answer: Clearly 0 is an upper bound for the sequence $\left( \frac{-1}{n} \right)_{n=1}^\infty$. It is the least upper bound of the sequence $\left( \frac{-1}{n} \right)_{n=1}^\infty$ because if $b < 0$ is a real number, then by the Archimedean Principle, there exists a positive integer $N$ such that $N > \frac{-1}{b}$ and so $\frac{-1}{N} > b$ which implies that $b < 0$ is not an upper bound of the sequence $\left( \frac{-1}{n} \right)_{n=1}^\infty$.

17. FALSE A sequence $(a_n)_{n=1}^\infty$ is said to converge to a real number $L$ if there exists a real number $\epsilon > 0$ and $N \in \mathbb{Z}^+$ such that for all $n \in \mathbb{Z}^+$, 
\[n > N \rightarrow |a_n - L| < \epsilon.\]

Answer: The definition of convergence of sequences is a very important fundamental concept that you shall understand; the correct definition is as follows: A sequence $(a_n)_{n=1}^\infty$ is said to converge to a real number $L$ if for every real number $\epsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that for all $n \in \mathbb{Z}^+$, 
\[n > N \rightarrow |a_n - L| < \epsilon.\]

In symbolic form, a sequence $(a_n)_{n=1}^\infty$ is said to converge to a real number $L$ if:
\[\forall \epsilon > 0 \exists N \in \mathbb{Z}^+ \forall n \in \mathbb{Z}^+ \ (n > N \rightarrow |a_n - L| < \epsilon).\]
18. **FALSE** The sequence \( \left( \frac{1}{n} \right)_{n=1}^\infty \) converges to 1.

**Answer:** The sequence \( \left( \frac{1}{n} \right)_{n=1}^\infty \) converges to 0. See the lecture notes for its proof using the definition of convergence of sequences and Archimedeian Principle.

19. **TRUE** Let \( c \in \mathbb{R} \). Let \( a_n = c \) for every \( n \in \mathbb{Z}^+ \). Then the constant sequence \( (a_n)_{n=1}^\infty = (c)_{n=1}^\infty \) converges to \( c \).

**Answer:** See the lecture notes for its proof. It is very clear by using the definition of convergence.

20. **FALSE** A sequence \( (a_n)_{n=1}^\infty \) may converge to a real number \( L_1 \) and to also another real number \( L_2 \) such that \( L_2 \neq L_1 \).

**Answer:** We must have \( L_1 = L_2 \). See the lecture notes for its proof. If a sequence \( (a_n)_{n=1}^\infty \) is convergent to a real number \( L \) and to a real number \( L_2 \), then we have proved in the lectures that \( L_1 = L_2 \). So there exists a unique real number \( L \) such that \( (a_n)_{n=1}^\infty \) converges to \( L \) and we call this unique real number \( L \) the limit of the convergent sequence \( (a_n)_{n=1}^\infty \) and we write \( \lim_{n \to \infty} a_n = L \).

21. **FALSE** A sequence \( (a_n)_{n=1}^\infty \) is said to be a convergent sequence if for every real number \( \varepsilon > 0 \), there exists \( N \in \mathbb{Z}^+ \) such that for all \( n \in \mathbb{Z}^+ \),

\[
N > n \implies |a_n - L| < \varepsilon.
\]

**Answer:** It is false since it does not state what \( L \) is; there is no quantifier for \( L \). The correct definition for convergent sequence must start with ‘there exists \( L \in \mathbb{R} \).’ The definition of convergent sequences is a very important fundamental concept that you shall understand; the correct definition is as follows: A sequence \( (a_n)_{n=1}^\infty \) is said to be convergent if there exists a real number \( L \) such that \( (a_n)_{n=1}^\infty \) converges to \( L \), that is, there exists a real number \( L \) such that for every real number \( \varepsilon > 0 \), there exists \( N \in \mathbb{Z}^+ \) such that for all \( n \in \mathbb{Z}^+ \),

\[
n > N \implies |a_n - L| < \varepsilon.
\]

In symbolic form, a sequence \( (a_n)_{n=1}^\infty \) is said to be a convergent sequence if:

\[
\exists L \in \mathbb{R} \quad \forall \varepsilon > 0 \quad \exists N \in \mathbb{Z}^+ \quad \forall n \in \mathbb{Z}^+ \quad (n > N \implies |a_n - L| < \varepsilon).
\]

22. **FALSE** A sequence \( (a_n)_{n=1}^\infty \) is said to be a divergent sequence if there exists \( \varepsilon > 0 \) and there exists \( N \in \mathbb{Z}^+ \) and \( L \in \mathbb{R} \) such that for all \( n \in \mathbb{Z}^+ \),

\[
n > N \quad \text{and} \quad |a_n - L| \geq \varepsilon.
\]

**Answer:** A sequence \( (a_n)_{n=1}^\infty \) is said to be divergent if it is not convergent, that is, if for every real number \( L \) there exists a real number \( \varepsilon > 0 \) such that for every \( N \in \mathbb{Z}^+ \), there exists \( n \in \mathbb{Z}^+ \) that satisfies

\[
n > N \quad \text{and} \quad |a_n - L| \geq \varepsilon.
\]

In symbolic form, a sequence \( (a_n)_{n=1}^\infty \) is said to be a divergent sequence if:

\[
\forall L \in \mathbb{R} \quad \exists \varepsilon > 0 \quad \forall N \in \mathbb{Z}^+ \quad \exists n \in \mathbb{Z}^+ \quad (n > N \quad \text{and} \quad |a_n - L| \geq \varepsilon).
\]

23. **FALSE** Every divergent sequence is unbounded.

**Answer:** For example, the sequence \( (-1)^n)_{n=1}^\infty \) is divergent and bounded.

24. **TRUE** Every subsequence of a convergent sequence is convergent.

**Answer:** See the lecture notes for its proof.

25. **FALSE** Every convergent sequence is bounded.

**Answer:** Every convergent sequence is bounded. See the lecture notes for its proof.

26. **FALSE** Every decreasing sequence is bounded.

**Answer:** For example, \( (-n)_{n=1}^\infty \) is a decreasing sequence which is not bounded.

27. **FALSE** Every unbounded sequence is monotone.

**Answer:** For example, the sequence \( ((-1)^n)_{n=1}^\infty \) is unbounded but it is neither nonincreasing nor nondecreasing, that is, it is not monotone as it is easily seen.

28. **TRUE** If \( (a_n)_{n=1}^\infty \) is an unbounded sequence, then \( (a_n)_{n=1}^\infty \) is divergent.

**Answer:** This is equivalent to saying that if \( (a_n)_{n=1}^\infty \) is convergent, then \( (a_n)_{n=1}^\infty \) is bounded, that is, every convergent sequence is bounded (see the lecture notes for its proof). The contrapositive of a statement “\( p \implies q \)” is “(not \( q \) ) \( \implies \) (not \( p \))” and this is logically equivalent to “\( \neg p \implies \neg q \)”.

29. **FALSE** A sequence \( (a_n)_{n=1}^\infty \) is said to be an increasing sequence if \( a_n < a_{n+1} \) for some \( n \in \mathbb{Z}^+ \).

**Answer:** A sequence \( (a_n)_{n=1}^\infty \) is said to be an increasing sequence if \( a_n < a_{n+1} \) for all \( n \in \mathbb{Z}^+ \).

30. **FALSE** A sequence \( (a_n)_{n=1}^\infty \) is said to be a nonincreasing sequence if \( a_n \leq a_{n+1} \) for all \( n \in \mathbb{Z}^+ \).

**Answer:** A sequence \( (a_n)_{n=1}^\infty \) is said to be a nonincreasing sequence if \( a_n \geq a_{n+1} \) for all \( n \in \mathbb{Z}^+ \).

31. **FALSE** A sequence \( (a_n)_{n=1}^\infty \) is said to be a decreasing sequence if \( a_n > a_{n+1} \) for some \( n \in \mathbb{Z}^+ \).

**Answer:** A sequence \( (a_n)_{n=1}^\infty \) is said to be a decreasing sequence if \( a_n > a_{n+1} \) for all \( n \in \mathbb{Z}^+ \).
32. **FALSE** A sequence \((a_n)_{n=1}^{\infty}\) is said to be a nondecreasing sequence if \(a_n \geq a_{n+1}\) for all \(n \in \mathbb{Z}^+\).

**Answer:** A sequence \((a_n)_{n=1}^{\infty}\) is said to be a nondecreasing sequence if \(a_n \leq a_{n+1}\) for all \(n \in \mathbb{Z}^+\).

33. **FALSE** A sequence \((a_n)_{n=1}^{\infty}\) is said to be a monotone sequence if it is an increasing sequence.

**Answer:** A sequence \((a_n)_{n=1}^{\infty}\) is said to be a monotone sequence if it is a nondecreasing sequence or a nonincreasing sequence.

Note that by the definitions of increasing, nonincreasing, decreasing and nondecreasing sequences, we have that: Increasing sequences are nondecreasing sequences but not conversely, and decreasing sequences are nonincreasing sequences but not conversely. Do you see the detail in these definitions?

34. **TRUE** An increasing sequence \((a_n)_{n=1}^{\infty}\) is always bounded from below.

**Answer:** If \((a_n)_{n=1}^{\infty}\) is an increasing sequence, then \(a_n < a_{n+1}\) for all \(n \in \mathbb{Z}^+\), and so

\[
a_1 < a_2 < a_3 < a_4 < \cdots a_n < a_{n+1} < \cdots
\]

which shows that \(a_1\) is a lower bound of the sequence \((a_n)_{n=1}^{\infty}\).

35. **TRUE** A decreasing sequence \((a_n)_{n=1}^{\infty}\) is always bounded from above.

**Answer:** If \((a_n)_{n=1}^{\infty}\) is a decreasing sequence, then \(a_n > a_{n+1}\) for all \(n \in \mathbb{Z}^+\), and so

\[
a_1 > a_2 > a_3 > a_4 > \cdots a_n > a_{n+1} < \cdots
\]

which shows that \(a_1\) is an upper bound of the sequence \((a_n)_{n=1}^{\infty}\).

36. **TRUE** Let \((a_n)_{n=1}^{\infty}\) be an increasing sequence. Then \((a_n)_{n=1}^{\infty}\) is a bounded sequence if and only if \((a_n)_{n=1}^{\infty}\) is bounded from above.

**Answer:** An increasing sequence is always bounded from below. So an increasing sequence \((a_n)_{n=1}^{\infty}\) is a bounded sequence if and only if \((a_n)_{n=1}^{\infty}\) is bounded from above.

37. **FALSE** Let \((a_n)_{n=1}^{\infty}\) be a decreasing sequence. Then \((a_n)_{n=1}^{\infty}\) is a bounded sequence if and only if \((a_n)_{n=1}^{\infty}\) is bounded from below.

**Answer:** A decreasing sequence is always bounded from above. So a decreasing sequence \((a_n)_{n=1}^{\infty}\) is a bounded sequence if and only if \((a_n)_{n=1}^{\infty}\) is bounded from below.

38. **FALSE** A nondecreasing sequence that is bounded from below is convergent.

**Answer:** A nondecreasing sequence is always bounded from below. Monotone Convergence Theorem for nondecreasing sequences says that a nondecreasing sequence bounded from below is convergent; it converges to the least upper bound of the sequence.

39. **FALSE** A nonincreasing sequence that is bounded from above is convergent.

**Answer:** A nonincreasing sequence is always bounded from above. Monotone Convergence Theorem for nonincreasing sequences says that a nonincreasing sequence bounded from below is convergent; it converges to the greatest lower of the sequence.

40. **FALSE** If \((a_n)_{n=1}^{\infty}\) is a monotone sequence, then \((a_n)_{n=1}^{\infty}\) is convergent.

**Answer:** A monotone sequence is convergent if and only if it is bounded. A monotone bounded sequence is convergent. This is the Monotone Convergence Theorem. See the lecture notes for its proof. For a counterexample, take \(a_n = n\) for all \(n \in \mathbb{Z}^+\). Then \((a_n)_{n=1}^{\infty}\) is an increasing sequence and so a monotone sequence but it is divergent.

41. **FALSE** If \((a_n)_{n=1}^{\infty}\) is bounded sequence, then \((a_n)_{n=1}^{\infty}\) is a convergent sequence.

**Answer:** For example, the sequence \((\{(-1)^n\}_{n=1}^{\infty})\) is a bounded sequence but it is divergent.

42. **TRUE** If \((a_n)_{n=1}^{\infty}\) is a decreasing sequence and \(a_n \geq 0\) for all \(n \in \mathbb{Z}^+\), then \((a_n)_{n=1}^{\infty}\) is convergent.

**Answer:** A decreasing sequence which is bounded from below is convergent by the Monotone Convergence Theorem. So the decreasing sequence \((a_n)_{n=1}^{\infty}\) that is bounded below by 0 is convergent.

43. **TRUE** Every bounded monotone sequence is convergent.

**Answer:** This is the Monotone Convergence Theorem. See the lecture notes for its proof.

44. **FALSE** If \(a_n \to A\) and \(b_n \to B\) for some real numbers \(A\) and \(B\), and if \(a_n > b_n\) for all \(n \in \mathbb{Z}^+\), then \(A > B\).

**Answer:** For example, for \(a_n = \frac{1}{n}\) and \(b_n = -\frac{1}{n}\) we have \(a_n > b_n\) for all \(n \in \mathbb{Z}^+\) but \(a_n \to 0\) and \(b_n \to 0\) so \(0 > 0\) is not true. We have in this case equality: \(0 = 0\). What we can say in general is that \(A \geq B\) (see the lecture notes for its proof), because we can have equality \(A = B\) as the example shows.

45. **TRUE** If \((a_n)_{n=1}^{\infty}\) converges to \(L\) and \((a_{n_k})_{k=1}^{\infty}\) is a subsequence of \((a_n)_{n=1}^{\infty}\), then \((a_{n_k})_{k=1}^{\infty}\) converges to \(L\), too.

**Answer:** See the lecture notes for its proof.

46. **TRUE** If \((a_n)_{n=1}^{\infty}\) has two convergent subsequences \((a_{n_k})_{k=1}^{\infty}\) and \((a_{m_n})_{n=1}^{\infty}\) such that \(\lim_{k \to \infty} a_{n_k} \neq \lim_{m \to \infty} a_{n_m}\), then \((a_n)_{n=1}^{\infty}\) is divergent.

**Answer:** If \((a_n)_{n=1}^{\infty}\) is convergent, then every subsequence has the same limit (namely the limit of the sequence \((a_n)_{n=1}^{\infty}\)). So if we have subsequences with different limits, then the original sequence cannot be convergent.

47. **FALSE** For all \(k \in \mathbb{Z}^+\), let \(n_k = \begin{cases} k^2, & \text{if } k \text{ is even}; \\ k, & \text{if } k \text{ is odd}. \end{cases}\)
Then \((a_n)_{n=1}^{\infty}\) is a subsequence of \((a_n)_{n=1}^{\infty}\).

**Answer:** By definition of subsequences of a sequence, the sequence \((n_k)_{k=1}^{\infty}\) of the indices for the subsequence must be an increasing sequence of positive integers, that is, \(n_k \in \mathbb{Z}^+\) and \(n_k < n_{k+1}\) for all \(k \in \mathbb{Z}^+\):

\[
n_1 < n_2 < n_3 < \cdots < n_k < n_{k+1} < \cdots
\]

In our case \(n_1 = 1 < n_2 = 2^2 > n_3 = 3 < n_4 = 4^2 > n_5 = 5 < n_6 = 6^2 > n_7 = 7 < \cdots\), that is,

\[
n_{2k-1} < n_{2k} \quad \text{and} \quad n_{2k} > n_{2k+1} \quad \text{for all} \quad k \in \mathbb{Z}^+.
\]

So the sequence \((n_k)_{k=1}^{\infty}\) of the indices is not increasing. Hence by the definition of subsequences, the sequence \((a_{n_k})_{k=1}^{\infty}\) is not a subsequence of \((a_n)_{n=1}^{\infty}\).

48. **TRUE** If \((a_n)_{n=1}^{\infty}\) converges to \(L\) and \((a_{n_k})_{k=1}^{\infty}\) is a subsequence of \((a_n)_{n=1}^{\infty}\), then \(n_k \geq k\) for all \(k \in \mathbb{Z}^+\).

**Answer:** By definition of subsequences of a sequence, the sequence \((n_k)_{k=1}^{\infty}\) of the indices for the subsequence must be an increasing sequence of positive integers, that is, \(n_k \in \mathbb{Z}^+\) and \(n_k < n_{k+1}\) for all \(k \in \mathbb{Z}^+\):

\[
n_1 < n_2 < n_3 < \cdots < n_k < n_{k+1} < \cdots
\]

Since \(n_k\) is a positive integer, we have \(n_1 \geq 1\). Since \(n_2 > n_1\) and \(n_1 \geq 1\), we must have \(n_2 \geq 2\). Since \(n_3 > n_2\) and \(n_2 \geq 2\), we must have \(n_3 \geq 3\). Continuing in this way one proves by induction on \(k\) that \(n_k \geq k\) for all \(k \in \mathbb{Z}^+\).

49. **TRUE** \(\lim_{n \to \infty} a_n = 0 \iff \lim_{n \to \infty} |a_n| = 0\).

**Answer:** See the lecture notes for its proof.

50. **FALSE** \(\lim_{n \to \infty} a_n = L \iff \lim_{n \to \infty} |a_n - L| = 0\).

**Answer:** See the lecture notes for its proof.

51. **FALSE** \(\lim_{n \to \infty} a_n = L \iff \lim_{n \to \infty} |a_n| = |L|\).

**Answer:** See the lecture notes for the proof of \(\lim_{n \to \infty} a_n = L \iff \lim_{n \to \infty} |a_n| = |L|\). The converse \(\lim_{n \to \infty} |a_n| = |L| \Rightarrow \lim_{n \to \infty} a_n = L\) is false since in general convergence of \(|a_n|\) does not imply convergence of \(a_n\). For example, if \(a_n = (−1)^n\) for all \(n \in \mathbb{Z}^+\), the sequence \([|a_n|]_{n=1}^{\infty} = (1)_{n=1}^{\infty}\) is the constant 1 sequence and so converges to 1 but \((a_n)_{n=1}^{\infty}\) is divergent.

52. **TRUE** If \(|a_n - L| \leq b_n\) for all \(n \in \mathbb{Z}^+\) beyond some index \(N_0\) and \(\lim_{n \to \infty} b_n = 0\), then \(\lim_{n \to \infty} a_n = L\).

**Answer:** See the lecture notes for its proof.

53. **FALSE** If \((a_n)_{n=1}^{\infty}\) is a divergent sequence, then \(|a_n|_{n=1}^{\infty}\) is also divergent.

**Answer:** If \(a_n = (−1)^n\) for all \(n \in \mathbb{Z}^+\), then \((a_n)_{n=1}^{\infty}\) is a divergent sequence but \(|a_n| = (−1)^n = 1\) for all \(n \in \mathbb{Z}^+\) and so \(|a_n|_{n=1}^{\infty}\) converges to 1.

54. **TRUE** If \(a_n \to 0\) and \((b_n)_{n=1}^{\infty}\) is bounded, then \(a_nb_n \to 0\).

**Answer:** See the lecture notes for its proof.

55. **FALSE** \(\lim \{a_nb_n\} = \lim \{a_n\} \cdot \lim \{b_n\}\) for all sequences \((a_n)_{n=1}^{\infty}\) and \((b_n)_{n=1}^{\infty}\).

**Answer:** This will be true if \((a_n)_{n=1}^{\infty}\) and \((b_n)_{n=1}^{\infty}\) are convergent sequences so that we have the right to write \(\lim \{a_n\}\) and \(\lim \{b_n\}\) which we know then to exist in real numbers. The limit theorem for the product of convergent sequences says that: If \((a_n)_{n=1}^{\infty}\) and \((b_n)_{n=1}^{\infty}\) are convergent sequences, then \((a_nb_n)_{n=1}^{\infty}\) is a convergent sequence and \(\lim \{a_nb_n\} = \lim \{a_n\} \cdot \lim \{b_n\}\).

56. **TRUE** If \((a_n)_{n=1}^{\infty}\) and \((b_n)_{n=1}^{\infty}\) are convergent sequences and \(a_n \leq b_n\) for all \(n \in \mathbb{Z}^+\), then \(\lim a_n \leq \lim b_n\).

**Answer:** See the lecture notes for its proof.

57. **FALSE** If \((a_n + b_n)_{n=1}^{\infty}\) is a convergent sequence, then both \((a_n)_{n=1}^{\infty}\) and \((b_n)_{n=1}^{\infty}\) are convergent.

**Answer:** For example, for \(a_n = (−1)^n\) and \(b_n = (−1)^{n+1}\) for all \(n \in \mathbb{Z}^+\), neither \((a_n)_{n=1}^{\infty}\) nor \((b_n)_{n=1}^{\infty}\) is convergent but \(a_n + b_n = 0\) for all \(n\), that is, \((a_n + b_n)_{n=1}^{\infty}\) is convergent.

58. **FALSE** If \(c \in \mathbb{R}\) and \((c_n)_{n=1}^{\infty}\) is a convergent sequence, then \((a_n+c_n)_{n=1}^{\infty}\) is a convergent sequence.

**Answer:** It is true if \(c \neq 0\) but for \(c = 0\), for example if \(a_n = (−1)^n\) for every \(n \in \mathbb{Z}^+\), then \((a_n)_{n=1}^{\infty}\) is divergent but \((c_n)_{n=1}^{\infty} = (0)_{n=1}^{\infty}\) is the constant zero sequence that converges to 0.

59. **FALSE** If \(a_nb_n \to 0\), then either \(a_n \to 0\) or \(b_n \to 0\).

**Answer:** For example, let \(a_n = 1\), if \(n\) is even; \(0\), if \(n\) is odd. and \(b_n = 1\), if \(n\) is even; \(0\), if \(n\) is odd.

Then \(a_nb_n \to 0\) but \(a_n \to 0\) and \(b_n \to 0\). Indeed, \((a_n)_{n=1}^{\infty}\) and \((b_n)_{n=1}^{\infty}\) are divergent.

60. **FALSE** If \((a_n+b_n)_{n=1}^{\infty}\) is a convergent sequence, then both \((a_n)_{n=1}^{\infty}\) and \((b_n)_{n=1}^{\infty}\) are convergent.

**Answer:** In the answer to the previous question, \((a_nb_n)_{n=1}^{\infty}\) is a convergent sequence (it converges to 0), but neither \((a_n)_{n=1}^{\infty}\) nor \((b_n)_{n=1}^{\infty}\) are convergent.

61. **FALSE** \(\lim \{c_n\} = c \lim \{a_n\}\) for every real number \(c\).

**Answer:** This will be true if \((a_n)_{n=1}^{\infty}\) is a convergent sequence so that we have the right to write \(\lim a_n\) which we know then to exist in real numbers. The limit theorem for this says that: If \((a_n)_{n=1}^{\infty}\) is a convergent sequence and \(c\) is a real number, then the sequence \((ca_n)_{n=1}^{\infty}\) is a convergent sequence and \(\lim \{ca_n\} = c \lim \{a_n\}\).
62. **FALSE** If \((a_n)_{n=1}^{\infty}\) is a convergent sequence, then the sequence \(\left(\frac{1}{a_n}\right)_{n=1}^{\infty}\) is also convergent.

**Answer:** For this to be true, \((a_n)_{n=1}^{\infty}\) must converge to a nonzero real number \(L\) and \(a_n \neq 0\) for all \(n \in \mathbb{Z}^+\) must hold. For example, if \(a_n = \frac{1}{n}\) for all \(n \in \mathbb{Z}^+\), then \((a_n)_{n=1}^{\infty}\) is a convergent sequence with limit 0, but \(\left(\frac{1}{a_n}\right)_{n=1}^{\infty} = (n)_{n=1}^{\infty}\) is a divergent sequence (it diverges to \(\infty\)).

63. **FALSE** If \((a_n)_{n=1}^{\infty}\) is a convergent sequence of nonzero real numbers with a nonzero limit, then the sequence \(\left(\frac{1}{a_n}\right)_{n=1}^{\infty}\) is bounded.

**Answer:** See the lecture notes for its proof. Indeed we have: if \((a_n)_{n=1}^{\infty}\) converge to a real number \(L \neq 0\) and \(a_n \neq 0\) for all \(n \in \mathbb{Z}^+\), then the sequence \(\left(\frac{1}{a_n}\right)_{n=1}^{\infty}\) converges to \(\frac{1}{L}\) and so it must be bounded. For the proof of this theorem, we firstly show that \(\left(\frac{1}{a_n}\right)_{n=1}^{\infty}\) is bounded and then \(\left|\frac{1}{a_n} - \frac{1}{L}\right| = \frac{1}{|a_n|}\left|a_n - L\right|\) is the product of a bounded sequence and a sequence with limit 0, and so converges to 0 which implies that \(\left(\frac{1}{a_n}\right)_{n=1}^{\infty}\) converges to \(\frac{1}{L}\).

64. **TRUE** If \(a_n \to L\) and if \(f(x)\) is a real-valued function of a real variable which is continuous at \(L\) and which is defined at \(a_n\) for all \(n \in \mathbb{Z}^+\), then the sequence \(\{f(a_n)\}_{n=1}^{\infty}\) converges to \(f(L)\).

**Answer:** See the lecture notes for its proof. This statement is half of the sequential characterization of continuity of a function at a point \(L\) in its domain.

65. **FALSE** A sequence \((a_n)_{n=1}^{\infty}\) is said to diverge to \(\infty\) if it is an increasing sequence that is not bounded from above.

**Answer:** A sequence \((a_n)_{n=1}^{\infty}\) is said to diverge to \(\infty\) if for every real number \(M\), there exists \(N \in \mathbb{Z}^+\) such that for all \(n \in \mathbb{Z}^+\),

\[n > N \implies a_n > M.\]

In symbolic form, a sequence \((a_n)_{n=1}^{\infty}\) is said to diverge to \(\infty\) if:

\[\forall M > 0 \exists N \in \mathbb{Z}^+ \forall n \in \mathbb{Z}^+ (n > N \implies a_n > M).\]

According to this definition, if a sequence \((a_n)_{n=1}^{\infty}\) diverges to \(\infty\), then it is necessarily true that it is not bounded from above but it may not be increasing. For example, for all \(n \in \mathbb{Z}^+\), let \(a_n = \begin{cases} n^2, & \text{if } n \text{ is even;} \\ n, & \text{if } n \text{ is odd.} \end{cases}\)

Then \((a_n)_{n=1}^{\infty}\) diverges to \(\infty\) but it is not increasing.

66. **FALSE** A sequence \((a_n)_{n=1}^{\infty}\) is said to diverge to \(-\infty\) if it is a decreasing sequence that is not bounded from below.

**Answer:** A sequence \((a_n)_{n=1}^{\infty}\) is said to diverge to \(-\infty\) if for every real number \(M\), there exists \(N \in \mathbb{Z}^+\) such that for all \(n \in \mathbb{Z}^+\),

\[n > N \implies a_n < M.\]

In symbolic form, a sequence \((a_n)_{n=1}^{\infty}\) is said to diverge to \(-\infty\) if:

\[\forall M > 0 \exists N \in \mathbb{Z}^+ \forall n \in \mathbb{Z}^+ (n > N \implies a_n < M).\]

According to this definition, if a sequence \((a_n)_{n=1}^{\infty}\) diverges to \(-\infty\), then it is necessarily true that it is not bounded from below but it may not be decreasing. For example, for all \(n \in \mathbb{Z}^+\), let \(a_n = \begin{cases} -n^2, & \text{if } n \text{ is even;} \\ -n, & \text{if } n \text{ is odd.} \end{cases}\)

Then \((a_n)_{n=1}^{\infty}\) diverges to \(-\infty\) but it is not decreasing.

67. **FALSE** If \((a_n)_{n=1}^{\infty}\), \((b_n)_{n=1}^{\infty}\) and \((c_n)_{n=1}^{\infty}\) are sequences of real numbers that satisfy

\[a_n \leq b_n \leq c_n \quad \text{for all } n \in \mathbb{Z}^+,\]

and if \((a_n)_{n=1}^{\infty}\) and \((c_n)_{n=1}^{\infty}\) are convergent, then \((b_n)_{n=1}^{\infty}\) is also convergent.

**Answer:** Do not mix this with Sandwich theorem; in Sandwich theorem, \((a_n)_{n=1}^{\infty}\) and \((c_n)_{n=1}^{\infty}\) have the same limit. Sandwich theorem says that: If \((a_n)_{n=1}^{\infty}\), \((b_n)_{n=1}^{\infty}\) and \((c_n)_{n=1}^{\infty}\) are sequences of real numbers that satisfy

\[a_n \leq b_n \leq c_n \quad \text{for all } n \in \mathbb{Z}^+,\]

and if for some real number \(L\), \((a_n)_{n=1}^{\infty}\) and \((c_n)_{n=1}^{\infty}\) both converge to the same real number \(L\), then \((b_n)_{n=1}^{\infty}\) converges to \(L\), too. In the case when \((a_n)_{n=1}^{\infty}\) and \((c_n)_{n=1}^{\infty}\) are convergent but not to the same real number, then \((b_n)_{n=1}^{\infty}\) may not be convergent even. For example, for all \(n \in \mathbb{Z}^+\), let \(a_n = -5\), \(b_n = (-1)^n\) and \(c_n = 3\). Then \(a_n \leq b_n \leq c_n\) for all \(n \in \mathbb{Z}^+\), \((a_n)_{n=1}^{\infty}\) converges to \(-5\), \((c_n)_{n=1}^{\infty}\) converges to 3 but \((b_n)_{n=1}^{\infty}\) is divergent.

68. **TRUE** If \((a_n)_{n=1}^{\infty}\) is an increasing sequence of real numbers, then either \((a_n)_{n=1}^{\infty}\) diverges to \(\infty\) (if it is not bounded from above), or \((a_n)_{n=1}^{\infty}\) is bounded from above and converges to \(\sup\{(a_n | n \in \mathbb{Z}^+)\}\).

**Answer:** See the lecture notes for its proof. It states the content of Monotone Convergence Theorem for increasing sequences: An increasing sequence \((a_n)_{n=1}^{\infty}\) of real numbers either diverges to \(\infty\) (if it is not bounded from above) or converges to \(L = \sup\{(a_n | n \in \mathbb{Z}^+)\}\) (if it is bounded from above).
69. **TRUE** If \((a_n)_{n=1}^{\infty}\) is a decreasing sequence of real numbers, then either \((a_n)_{n=1}^{\infty}\) diverges to \(-\infty\) (if it is not bounded from below), or \((a_n)_{n=1}^{\infty}\) is bounded from below and converges to \(\inf\{a_n \mid n \in \mathbb{Z}^+\}\).

**Answer:** See the lecture notes for its proof. It states the content of Monotone Convergence Theorem for decreasing sequences: A decreasing sequence \((a_n)_{n=1}^{\infty}\) of real numbers either diverges to \(-\infty\) (if it is not bounded from below) or converges to \(L = \inf\{a_n \mid n \in \mathbb{Z}^+\}\) (if it is bounded from below).

70. **FALSE** \(\lim_{n \to \infty} (a_n + b_n) = \left[ \lim_{n \to \infty} a_n \right] + \left[ \lim_{n \to \infty} b_n \right]\) for all sequences \((a_n)_{n=1}^{\infty}\) and \((b_n)_{n=1}^{\infty}\).

**Answer:** This will be true if \((a_n)_{n=1}^{\infty}\) and \((b_n)_{n=1}^{\infty}\) are convergent sequences so that we have the right to write \(\lim_{n \to \infty} a_n\) and \(\lim_{n \to \infty} b_n\) which we know then to exist in real numbers. The limit theorem for the sum of convergent sequences says that: If \((a_n)_{n=1}^{\infty}\) and \((b_n)_{n=1}^{\infty}\) are convergent sequences, then \((a_n + b_n)_{n=1}^{\infty}\) is a convergent sequence and \(\lim_{n \to \infty} (a_n + b_n) = \left[ \lim_{n \to \infty} a_n \right] + \left[ \lim_{n \to \infty} b_n \right]\).

71. **TRUE** Let \(f\) be a function defined on the interval \([N_0, \infty)\) for some \(N_0 \in \mathbb{Z}^+\) and let \((a_n)_{n=1}^{\infty}\) be a sequence of real numbers such that:

\[ a_n = f(n) \quad \text{for all integers } n \geq N_0. \]

If \(\lim_{x \to \infty} f(x) = L\) for some real number \(L\), then \((a_n)_{n=1}^{\infty}\) converges to \(L\), that is, \(\lim_{n \to \infty} a_n = L\).

**Answer:** See the lecture notes for the proof of this theorem. It is like the previous question with \(\infty\) in place of \(L\).

72. **TRUE** Let \(f\) be a function defined on the interval \([N_0, \infty)\) for some \(N_0 \in \mathbb{Z}^+\) and let \((a_n)_{n=1}^{\infty}\) be a sequence of real numbers such that:

\[ a_n = f(n) \quad \text{for all integers } n \geq N_0. \]

If \(\lim_{x \to \infty} f(x) = \infty\), then \((a_n)_{n=1}^{\infty}\) diverges to \(\infty\), that is, \(\lim_{n \to \infty} a_n = \infty\).

**Answer:** See the lecture notes for the proof of this theorem. It is like the previous question with \(-\infty\) in place of \(\infty\).

73. **TRUE** Let \(f\) be a function defined on the interval \([N_0, \infty)\) for some \(N_0 \in \mathbb{Z}^+\) and let \((a_n)_{n=1}^{\infty}\) be a sequence of real numbers such that:

\[ a_n = f(n) \quad \text{for all integers } n \geq N_0. \]

If \(\lim_{x \to \infty} f(x) = -\infty\), then \((a_n)_{n=1}^{\infty}\) diverges to \(-\infty\), that is, \(\lim_{n \to \infty} a_n = -\infty\).

**Answer:** See the lecture notes for the proof of this theorem. It is like the previous question with \(-\infty\) in place of \(\infty\).

74. **FALSE** Let \(f\) be a function defined on the interval \([N_0, \infty)\) for some \(N_0 \in \mathbb{Z}^+\) and let \((a_n)_{n=1}^{\infty}\) be a sequence of real numbers such that:

\[ a_n = f(n) \quad \text{for all integers } n \geq N_0. \]

If the sequence \((a_n)_{n=1}^{\infty}\) converges to \(L\), that is, \(\lim_{n \to \infty} a_n = L\) for some real number \(L\), then we also have that \(\lim_{x \to \infty} f(x) = L\).

**Answer:** Let \(f(x) = \sin(2\pi x)\) for all \(x \in \mathbb{R}\) and \(a_n = 0\) for all \(n \in \mathbb{Z}^+\). Then \(a_n = f(n)\) for all \(n \in \mathbb{Z}^+\), and the sequence \((a_n)_{n=1}^{\infty}\) converges to 0 since it is the constant zero sequence. But \(\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \sin(2\pi x)\) does not exist since the function \(f(x) = \sin(2\pi x)\) oscillates infinitely many times as \(x \to \infty\); consider the graph of the function \(f(x) = \sin(2\pi x)\) to see that its limit as \(x \to \infty\) does not exist.

75. **TRUE** If \((a_n)_{n=1}^{\infty}\) is a divergent sequence and \(c\) a nonzero real number, then the sequence \((ca_n)_{n=1}^{\infty}\) is divergent, too.

**Answer:** If \((ca_n)_{n=1}^{\infty}\) were a convergent sequence, then since \(c \neq 0\), the sequence \(\left\{ \frac{1}{c} \cdot ca_n \right\} = (a_n)_{n=1}^{\infty}\) would be a divergent sequence which contradicts with the hypothesis that \((a_n)_{n=1}^{\infty}\) is a divergent sequence.

Indeed, for a sequence \((a_n)_{n=1}^{\infty}\) of real numbers and for a nonzero real number \(c\), prove that:

\[ (a_n)_{n=1}^{\infty} \text{ is a convergent sequence} \iff (ca_n)_{n=1}^{\infty} \text{ is a convergent sequence} \]

So for a sequence \((a_n)_{n=1}^{\infty}\) of real numbers and for a nonzero real number \(c\), we have that:

\[ (a_n)_{n=1}^{\infty} \text{ is a divergent sequence} \iff (ca_n)_{n=1}^{\infty} \text{ is a divergent sequence} \]

76. **TRUE** If \((a_n)_{n=1}^{\infty}\) is a convergent sequence, then \(\lim_{n \to \infty} (-a_n) = -\left[ \lim_{n \to \infty} a_n \right]\).

**Answer:** See the lecture notes for the proof of the following theorem: If \((a_n)_{n=1}^{\infty}\) is a convergent sequence, then for every real number \(c\), \((ca_n)_{n=1}^{\infty}\) is a convergent sequence with \(\lim_{n \to \infty} (-a_n) = -\left[ \lim_{n \to \infty} a_n \right]\). For \(c = -1\), we obtain that if \((a_n)_{n=1}^{\infty}\) is a convergent sequence, then \((-a_n)_{n=1}^{\infty}\) is a convergent sequence with \(\lim_{n \to \infty} (-a_n) = -\left[ \lim_{n \to \infty} a_n \right]\).
77. **FALSE** \( \lim_{n \to \infty} (a_n - b_n) = \lim_{n \to \infty} a_n - \lim_{n \to \infty} b_n \) for all sequences \((a_n)_{n=1}^{\infty}\) and \((b_n)_{n=1}^{\infty}\).

**Answer:** This will be true if \((a_n)_{n=1}^{\infty}\) and \((b_n)_{n=1}^{\infty}\) are convergent sequences so that we have the right to write \( \lim_{n \to \infty} a_n \) and \( \lim_{n \to \infty} b_n \), which we know then to exist in real numbers. The limit theorem for the difference of convergent sequences says that if \((a_n)_{n=1}^{\infty}\) and \((b_n)_{n=1}^{\infty}\) are convergent sequences, then \((a_n - b_n)_{n=1}^{\infty}\) is a convergent sequence and
\[
\lim_{n \to \infty} (a_n - b_n) = \lim_{n \to \infty} a_n - \lim_{n \to \infty} b_n.
\]
This just follows from the theorem for the sum of convergent sequences and by the theorem in the previous question.

78. **TRUE** Let \( f \) be a function defined on \([d, \infty)\) for some real number \( d \). Let \((a_n)_{n=1}^{\infty}\) be a sequence of real numbers that diverges to \( \infty \). If \( \lim_{x \to \infty} f(x) = L \) for some real number \( L \), and if \( a_n \in [d, \infty) \) for all \( n \in \mathbb{N}^+ \), then the sequence \((f(a_n))_{n=1}^{\infty}\) converges to \( L \), that is, \( \lim_{n \to \infty} f(a_n) = L \).

**Answer:** Prove this by using the definition of \( \lim_{x \to \infty} f(x) = L \) and \( \lim_{n \to \infty} a_n = \infty \).

79. **TRUE** Let \( L \in \mathbb{R} \). Let \( f \) be a function that is defined at \( L \). If \( f \) is not continuous at \( L \), then there exists a sequence \((a_n)_{n=1}^{\infty}\) that converges to \( L \) such that \( a_n \) is in Domain(\( f \)) for every \( n \in \mathbb{N}^+ \) but \((f(a_n))_{n=1}^{\infty}\) does not converge to \( f(L) \) (that is, either \((f(a_n))_{n=1}^{\infty}\) is a divergent sequence or \((f(a_n))_{n=1}^{\infty}\) is a convergent sequence but its limit is not equal to \( f(L) \)).

**Answer:** Prove this using the \( \epsilon-\delta \) definition of continuity and convergence of sequences. It is part of the theorem for Sequential Characterization of Continuity.

80. **TRUE** If \((a_n)_{n=1}^{\infty}\) is a sequence of real numbers such that there exists a real number \( L \) and both of the subsequences \((a_{2k-1})_{k=1}^{\infty}\) and \((a_{2k})_{k=1}^{\infty}\) converge to \( L \), then \((a_n)_{n=1}^{\infty}\) converges to \( L \).

**Answer:** Prove this using the definition of convergence of sequences.

81. **FALSE** If \( a_n \neq 0 \) for all \( n \in \mathbb{Z}^+ \) and \( \frac{1}{a_n} \to 0 \), then \( a_n \to \infty \).

**Answer:** For example, for \( a_n = (-1)^n n \), we have \( a_n \neq 0 \) for all \( n \in \mathbb{Z}^+ \) and \( \frac{1}{a_n} \to 0 \). But \( a_n \nrightarrow \infty \). Indeed, the following is true: If \( a_n \neq 0 \) for all \( n \in \mathbb{Z}^+ \) and \( \frac{1}{a_n} \to 0 \), then \( |a_n| \to \infty \). So it is true that: If \( a_n > 0 \) for all \( n \in \mathbb{Z}^+ \) and \( \frac{1}{a_n} \to 0 \), then \( a_n \to \infty \).

82. **TRUE** If \( a_n > 0 \) for all \( n \in \mathbb{Z}^+ \) and \( \frac{1}{a_n} \to 0 \), then \( a_n \to \infty \).

**Answer:** Prove this.

83. **FALSE** If \( a_n < 0 \) for all \( n \in \mathbb{Z}^+ \) and \( \frac{1}{a_n} \to 0 \), then \( a_n \to \infty \).

**Answer:** In this case \( a_n \to -\infty \). That is, prove that if \( a_n < 0 \) for all \( n \in \mathbb{Z}^+ \) and \( \frac{1}{a_n} \to 0 \), then \( a_n \to -\infty \).

84. **TRUE** Let \( p \in \mathbb{R} \). Then \( \lim_{n \to \infty} \frac{1}{n^p} = 0 \) if and only if \( p > 0 \).

**Answer:** See the lecture notes for its proof. If \( p > 0 \), it is shown that \( \lim_{n \to \infty} \frac{1}{n^p} = 0 \). If \( p = 0 \) \( \lim_{n \to \infty} \frac{1}{n^p} = 1 \). If \( p < 0 \), then \( \left( \frac{1}{n^p} \right)_{n=1}^{\infty} = (n^{-p})_{n=1}^{\infty} \) is an unbounded increasing sequence, so it is divergent.

85. **FALSE** For a positive real number \( a \), \( \lim_{n \to \infty} a^{1/n} = 0 \).

**Answer:** See the lecture notes for the proof of \( \lim_{n \to \infty} a^{1/n} = 1 \).

86. **FALSE** \( \lim_{n \to \infty} \frac{5^n}{n!} = 1 \).

**Answer:** Remember that we have proved in the lectures that for every real number \( x \), \( \lim_{n \to \infty} \frac{x^n}{n!} = 0 \). So for \( x = 5 \), we obtain \( \lim_{n \to \infty} \frac{5^n}{n!} = 0 \).

87. **FALSE** \( \lim_{n \to \infty} \sqrt[n]{n} = \infty \).

**Answer:** See the lecture notes for the proof of \( \lim_{n \to \infty} \sqrt[n]{n} = 1 \).

88. **FALSE** \( \lim_{n \to \infty} \frac{n}{2^n} = 1 \).

**Answer:** By the next question with \( a = 2 > 1 \) and \( p = 1 \), we have \( \lim_{n \to \infty} \frac{n}{2^n} = 0 \).

89. **TRUE** \( \lim_{n \to \infty} a^n = 0 \) if \( a > 1 \) is a real number and \( p \in \mathbb{R} \) is arbitrary.

**Answer:** See the lecture notes for its proof.

90. **FALSE** \( \lim_{n \to \infty} \frac{\ln(n)}{n^c} = 1 \) for every positive real number \( c \).

**Answer:** See the lecture notes for the proof of \( \lim_{n \to \infty} \frac{\ln(n)}{n^c} = 0 \). Similar to that proof obtain that \( \lim_{n \to \infty} \frac{\ln(n)}{n^c} = 0 \) for every positive real number \( c \).
91. **FALSE** Let \( r \in \mathbb{R} \). Then the sequence \((r^n)_{n=1}^{\infty}\) is convergent if and only if \(|r| > 1\) or \( r = 1 \).

**Answer:** \((r^n)_{n=1}^{\infty}\) is convergent if and only if \(|r| < 1\) or \( r = 1 \) (see the lecture notes for the proof).

92. **FALSE** If \(|r| < 1\), then \( \lim_{n \to \infty} r^n = 1 \).

**Answer:** If \(|r| < 1\), then \( r^n \to 0 \). See the lecture notes for the proof of this frequently used limit.

93. **FALSE** \( \lim_{n \to \infty} \frac{n!}{10^n} = 1 \).

**Answer:** Remember that we have proved in the lectures that for every real number \( x \), \( \lim_{n \to \infty} \frac{x^n}{n!} = 0 \). So for \( x = 10 \), we obtain \( \lim_{n \to \infty} \frac{10^n}{n!} = 0 \). Since \( \frac{n!}{10^n} > 0 \) for all \( n \in \mathbb{Z}^+ \), we obtain \( \lim_{n \to \infty} \frac{n!}{10^n} = \lim_{n \to \infty} \frac{1}{10^n/n!} = 0 \).

94. **FALSE** \( \lim_{n \to \infty} \left( 1 - \frac{1}{n} \right)^n = e \).

**Answer:** Remember that we have proved in the lectures that for every real number \( x \), \( \lim_{n \to \infty} \left( 1 + \frac{x}{n} \right)^n = e^x \). So for \( x = -1 \), we obtain \( \lim_{n \to \infty} \left( 1 - \frac{1}{n} \right)^n = \lim_{n \to \infty} \left( 1 + \frac{-1}{n} \right)^n = e^{-1} \).

95. **FALSE** The sequence \( \left( \frac{\sin n}{n^2} \right)_{n=1}^{\infty} \) is a divergent sequence.

**Answer:** Since \(|\sin n| \leq 1\), that is, \(-1 \leq \sin n \leq 1\) for all \( n \in \mathbb{Z}^+ \), we have \( \frac{-1}{n^2} \leq \frac{\sin n}{n^2} \leq \frac{1}{n^2} \) for all \( n \in \mathbb{Z}^+ \). Since \( \lim_{n \to \infty} \frac{-1}{n^2} = 0 \), \( \lim_{n \to \infty} \frac{\sin n}{n^2} = 0 \), we obtain by the Sandwich Theorem that the sequence \( \left( \frac{\sin n}{n^2} \right)_{n=1}^{\infty} \) is a convergent sequence with \( \lim_{n \to \infty} \frac{\sin n}{n} = 0 \).

96. **TRUE** Let \( (a_n)_{n=1}^{\infty} \) be a sequence of positive real numbers:

\[ a_n > 0 \quad \text{for all } n \in \mathbb{Z}^+. \]

For every \( n \in \mathbb{Z}^+ \), let

\[ s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots + a_n. \]

Then \( (s_n)_{n=1}^{\infty} \) is an increasing sequence.

**Answer:** This is clear since \( a_n > 0 \) for all \( n \in \mathbb{Z}^+ \). Because for all \( n \in \mathbb{Z}^+ \),

\[ s_{n+1} = \sum_{k=1}^{n+1} a_k = a_1 + a_2 + \ldots + a_n + a_{n+1} = \sum_{k=1}^{n} a_k + a_{n+1} = s_n + a_{n+1} > s_n \quad \text{since } a_{n+1} > 0. \]

That is, \( s_{n+1} > s_n \) for all \( n \in \mathbb{Z}^+ \) which means that \( (s_n)_{n=1}^{\infty} \) is an increasing sequence.

97. **FALSE** Let \( (a_n)_{n=1}^{\infty} \) be a sequence of positive real numbers:

\[ a_n > 0 \quad \text{for all } n \in \mathbb{Z}^+. \]

For every \( n \in \mathbb{Z}^+ \), let

\[ s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots + a_n. \]

Then \( (s_n)_{n=1}^{\infty} \) is convergent.

**Answer:** By the previous question, \( (s_n)_{n=1}^{\infty} \) is an increasing sequence. By the Monotone Convergence Theorem, \( (s_n)_{n=1}^{\infty} \) is convergent if and only if \( (s_n)_{n=1}^{\infty} \) is a bounded sequence. So it is not always true that \( (s_n)_{n=1}^{\infty} \) is convergent. For a very simple example, take \( a_n = 1 \) for all \( n \in \mathbb{Z}^+ \). Then for every \( n \in \mathbb{Z}^+ \), \( s_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} 1 = n \), and so the sequence \( (s_n)_{n=1}^{\infty} = (n)_{n=1}^{\infty} \) diverges to \( \infty \), it is not convergent.

98. **FALSE** If \( a_n \to L \) and \( b_n \to \infty \), then \( a_n b_n \to \infty \).

**Answer:** For example, if \( a_n = 0 \) for all \( n \) and \( b_n = n \) for all \( n \), then \( a_n \to 0 \) and \( b_n \to \infty \) but \( a_n b_n \to 0 \).

99. **TRUE** If \( a_n \to 0 \), \( b_n \to 1 \), then \( a_n b_n \to 0 \).

**Answer:** Since \( b_n \to 1 \), \( (b_n)_{n=1}^{\infty} \) is a bounded sequence. We have more generally that if \( a_n \to 0 \) and \( (b_n)_{n=1}^{\infty} \) is bounded, then \( a_n b_n \to 0 \) (see the lecture notes for its proof). Or use the theorem for the convergence of the product of convergent sequences: if \( a_n \to L \) for some real number \( L \) and \( a_n \to M \) for some real number \( M \), then \( a_n b_n \to LM \).

100. **FALSE** If \( (a_n)_{n=1}^{\infty} \) is divergent, then \( (a_n^2)_{n=1}^{\infty} \) is divergent.

**Answer:** For example, for \( a_n = (-1)^n \), \( (a_n^2)_{n=1}^{\infty} \) is divergent, but \( (a_n^2)_{n=1}^{\infty} \) is convergent since \( a_n^2 = 1 \) for all \( n \).

101. **FALSE** If \( (a_n)_{n=1}^{\infty} \) is convergent and \( (b_n)_{n=1}^{\infty} \) is bounded, then \( (a_n b_n)_{n=1}^{\infty} \) is convergent.

**Answer:** For example, for \( a_n = 1 \) for all \( n \) and \( b_n = (-1)^n \) for all \( n \), \( (a_n)_{n=1}^{\infty} \) is convergent and \( (b_n)_{n=1}^{\infty} \) is bounded but \( (a_n b_n)_{n=1}^{\infty} \) is divergent since \( a_n b_n = (-1)^n \) for all \( n \).
102. **FALSE** If \( a_n \to 0 \) and \( b_n \to L \), then \( \frac{a_n}{b_n} \to 0 \).

**Answer:** For example, let \( a_n = \frac{1}{n} = b_n \) for all \( n \in \mathbb{Z^+} \). Then \( a_n \to 0 \) and \( b_n \to 0 \) (so \( L = 0 \) in this case) but \( \frac{a_n}{b_n} \to 1 \).

103. **TRUE** If \( a_n \to L \) and \( b_n \to \infty \), then \( a_n + b_n \to \infty \).

**Answer:** See the lecture notes for its proof.

104. **FALSE** If \( a_n \to \infty \) and \( \{b_n\}_{n=1}^\infty \) is bounded, then \( a_n b_n \to \infty \).

**Answer:** For example, for \( a_n = n \) and \( b_n = 0 \) for all \( n \), \( a_n \to \infty \) and \( \{b_n\}_{n=1}^\infty \) is bounded but \( a_n b_n = 0 \to \infty \).

105. **TRUE** If \( a_n \to \infty \) or \( a_n \to -\infty \), then \( |a_n| \to \infty \).

**Answer:** See the lecture notes for its proof.

106. **FALSE** If \( |a_n| \to \infty \) and \( b_n \geq a_n \) for all \( n \in \mathbb{Z^+} \), then \( b_n \to \infty \).

**Answer:** Let \( a_n = b_n = (-1)^n n \). Then \( |a_n| = n \to \infty \) and \( b_n = a_n \geq a_n \) for all \( n \in \mathbb{Z^+} \). But \( b_n \not\to \infty \).

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