NORMAL AND TANGENTIAL COORDINATES (n-t)
One of the common descriptions of curvilinear motion uses **path variables**, which are measurements made along the tangent $t$ and normal $n$ to the path of the particles.

These coordinates provide a very natural description for curvilinear motion and are frequently the most direct and convenient coordinates to use and move along the path with the particle.

The positive direction for $n$ at any position is always taken toward the center of curvature of the path.
We now use the coordinates $n$ and $t$ to describe the velocity and acceleration. For this purpose, we introduce unit vectors $\vec{e}_n$ in the $n$-direction and $\vec{e}_t$ in the $t$-direction.

During a differential increment of time $dt$, the particle moves a differential distance $ds$ along the curve from $A$ to $A'$.

$$ds = \rho d\beta$$ where $\beta$ is in radians.

It is unnecessary to consider the differential change in $\rho$ between $A$ and $A'$.

$$v = \frac{ds}{dt} = \rho \frac{d\beta}{dt} = \rho \dot{\beta}$$

$$\vec{v} = \rho \dot{\beta} \vec{e}_t$$
ACCELERATION

The acceleration of the particle was defined as \( \vec{a} = \frac{d\vec{v}}{dt} \), and we observe that the acceleration is a vector which reflects both the change in magnitude and the change in direction of velocity.

We now differentiate the velocity applying the ordinary rule for the differentiation of the product of a scalar and a vector and get

\[
\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(v\vec{e}_t) = \dot{v}\vec{e}_t + v\dot{\vec{e}}_t
\]

The unit vector \( \vec{e}_t \) now has a nonzero derivative because its directions changes.
To find \( \dot{\bar{e}}_t \), we analyze the change in \( \bar{e}_t \) during a differential increment of motion as the particle moves from A to A'.

\[
\dot{\bar{e}}_t = \frac{\ddot{\bar{e}}_t}{dt} = \frac{\ddot{\bar{e}}_t}{d\beta} \cdot \frac{d\beta}{dt} \quad \frac{|d\bar{e}_t|}{d\beta} = ?
\]

\[
\sin \frac{d\beta}{2} = \frac{|d\bar{e}_t|/2}{|d\bar{e}_t|} \quad \frac{|d\bar{e}_t|}{d\beta} = 1
\]

The direction of \( d\bar{e}_t \) is given by \( \bar{e}_n \).

\[
\dot{\bar{e}}_t = \beta \bar{e}_n
\]
\[ \ddot{a} = \ddot{v} \hat{e}_t + \dot{v} \hat{e}_t = \ddot{v} \hat{e}_t + \dot{v} \hat{\beta} \hat{e}_n \]

\[ \ddot{a} = \ddot{v} \hat{e}_t + \dot{v} \hat{e}_t = \ddot{v} \hat{e}_t + \dot{v} \cdot \frac{v}{\rho} \hat{e}_n \]

\[ \ddot{a} = \ddot{v} \hat{e}_t + \dot{v} \hat{e}_t = \ddot{v} \hat{e}_t + \frac{v^2}{\rho} \hat{e}_n \]

\[ a_n = \frac{v^2}{\rho} = v \dot{\beta} = \rho \dot{\beta}^2 \]

\[ a_t = \dot{v} = \rho \ddot{\beta} = \dddot{s} \]

\[ a = \sqrt{a_t^2 + a_n^2} \]

(Change in direction of the velocity)

(Change in magnitude of the velocity)

Radius of curvature: If trajectory is given as \( y = f(x) \)

\[ \rho = \frac{1 + \left( \frac{dy}{dx} \right)^2}{\left| \frac{d^2y}{dx^2} \right|} \]
The normal component of acceleration $a_n$ is always directed toward the center of curvature. 
The tangential of acceleration $a_t$, on the other hand, will be in the positive $t$-direction of motion if speed $v$ is increasing and in the negative $t$-direction if the speed is decreasing.

Figure shows the schematic representations of the variation in the acceleration vector for a particle moving from A to B with (a) increasing speed and (b) decreasing speed.

At an inflection point on the curve, the normal acceleration goes to zero because $\rho$ becomes infinite.

$$a_n = \frac{v^2}{\rho} = \frac{v^2}{\infty} = 0$$
Circular motion is an important special case of plane curvilinear motion where the radius of curvature $\rho$ becomes the constant radius $r$ of the circle and the angle $\beta$ is replaced by the angle $\theta$ measured from any convenient radial reference to OP.

\[ \rho = r = \text{constant} \]

\[ a_n = \frac{v^2}{r} = r\dot{\theta}^2 = v\dot{\theta} \]

\[ a_t = \ddot{v} = r\ddot{\theta} \]

\[ \omega = \frac{d\theta}{dt} = \dot{\theta} \quad \text{(Angular velocity)} \]

\[ \alpha = \frac{d\omega}{dt} = \ddot{\theta} \quad \text{(Angular acceleration)} \]
1. Six acceleration vectors are shown for the car whose velocity vector is directed forward. For each acceleration vector describe in words the instantaneous motion of the car.
2. A train enters a curved horizontal section of track at a speed of 100 km/h and slows down with constant deceleration to 50 km/h in 12 seconds. An accelerometer mounted inside the train records a horizontal acceleration of 2 m/s² when the train is 6 seconds into the curve. Calculate the radius of curvature $\rho$ of the track for this instant.
3. The design of a camshaft-drive system of a four cylinder automobile engine is shown. As the engine is revved up, the belt speed $v$ changes uniformly from 3 m/s to 6 m/s over a two-second interval. Calculate the magnitudes of the accelerations of point $P_1$ and $P_2$ halfway through this time interval.
4. A ball is thrown horizontally from the top of a 50 m cliff at A with a speed of 15 m/s and lands at point C. Because of strong horizontal wind the ball has a constant acceleration in the negative x-direction. Determine the radius of curvature $\rho$ of the path of the ball at B where its trajectory makes an angle of $45^\circ$ with the horizontal. Neglect any effect of air resistance in the vertical direction.
5. The pin P is constrained to move in the slotted guides which move at right angles to one another. At the instant represented, A has a velocity to the right of 0.2 m/s which is decreasing at the rate of 0.75 m/s each second. At the same time, B is moving down with a velocity of 0.15 m/s which is decreasing at the rate of 0.5 m/s each second. For this instant, determine the radius of curvature $\rho$ of the path followed by P.
6. Pin P in the crank PO engages the horizontal slot in the guide C and controls its motion on the fixed vertical rod. Determine the velocity \( y \) and the acceleration \( \ddot{y} \) of guide C for a given value of the angle \( \theta \) is

(a) \( \dot{\theta} = \omega \) and \( \ddot{\theta} = 0 \)

(b) if \( \dot{\theta} = 0 \) and \( \ddot{\theta} = \alpha \)